1. (a) A small furniture company manufactures tables and chairs. Production is in two stages, assembly and finishing, the time taken for each product and available time per week being as follows.

	Table (hours)	Chair (hours)	Availability (hours/week)
Assembly	2	0.5	100
Finishing	1	0.5	120

Customers usually buy at least 4 chairs with each table, meaning that the factory must produce at least 4 times as many chairs as tables.

Tables sell for £250 each, while chairs sell for £50 each. Raw material costs are £30 for tables and £10 for chairs, while production costs are £20 per hour, whether for assembly or finishing. The company wishes to maximise its weekly profit.

Formulate this problem as a linear program. (Do **not** go on to solve it.)

[8 marks]

(b) Consider the linear program

maximise $x_0 = ax_1 + x_2$
subject to

$$\begin{aligned} 3x_1 + 2x_2 &\le b \\ 2x_1 + 3x_2 &\le 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

With $a = 1$ and $b = 6$, sketch the feasible region of this linear program.

Answer the following questions using your sketch. (Do **not** use tableaux.)

(i) With $a = 1$ and $b = 6$, find the optimal solution.

(ii) Keeping $a = 1$, by how much must b be increased before the constraint $3x_1 + 2x_2 \le b$ becomes redundant?

(iii) Keeping $b = 6$, for what range of values of a is the optimal solution unchanged?

(iv) Find the optimal solution if $a = -1$ and $b = 6$.

[12 marks]

2. (a) Use the primal simplex method to solve the linear program

maximise $x_0 = 2x_1 + x_2 - 3x_3 + 5x_4$
subject to

$$\begin{aligned} x_1 + 7x_2 + 3x_3 + 7x_4 &\leq 48 \\ 3x_1 - x_2 + x_3 + 2x_4 &\leq 8 \\ 2x_1 + 3x_2 - x_3 + x_4 &\leq 10 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

[10 marks]

(b) Use the dual simplex method to solve the following linear program. (Solution by any other method will **not** receive credit.)

maximise $x_0 = -3x_1 - x_2 - 3x_3$
subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 8 \\ x_1 + 2x_2 + x_3 &\geq 8 \\ 2x_1 + 4x_2 + x_3 &\geq 12 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

[10 marks]

3. (a) Let $\{x_0^*, x_1^*, \ldots, x_n^*\}$ and $\{y_0^*, y_1^*, \ldots, y_m^*\}$ denote optimal solutions of the following pair of primal and dual linear programs:

P: maximise $x_0 = p_1x_1 + p_2x_2 + \ldots + p_nx_n$
subject to

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \leq & b_2 \\ \cdots\cdots\cdots & \cdots & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \leq & b_m \\ x_1, x_2, \ldots, x_n & \geq & 0. \end{array}$$

D: minimise $y_0 = b_1y_1 + b_2y_2 + \cdots + b_my_m$
subject to

$$\begin{array}{rcl} a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m & \geq & p_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m & \geq & p_2 \\ \cdots\cdots\cdots & \cdots & \cdots \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m & \geq & p_n \\ y_1, y_2, \ldots, y_m & \geq & 0. \end{array}$$

State the *Duality Theorem*.

[6 marks]

(b) Write down the dual linear program D of
P: maximise $x_0 = 8x_1 + x_2$ subject to

$$\begin{array}{rcl} 5x_1 + 2x_2 & \leq & 10 \\ x_1 + 2x_2 & \leq & 4 \\ x_1, x_2 & \geq & 0 \end{array}$$

Solve both the primal and the dual graphically, and check that the complementary slackness conditions are satisfied.

[14 marks]

4. (a) For a minimisation bicriterion problem with objectives Z_1 and Z_2 say what it means for a point $\mathbf{y}$ to be *inferior to* a point $\mathbf{x}$.

   Sketch in **objective space** the set of points inferior to a given point $(Z_1(\mathbf{x}), Z_2(\mathbf{x}))$.

   [3 marks]

   (b) Consider the bicriterion problem
   minimise $\{Z_1, Z_2\}$
   subject to

$$\begin{aligned} x + 2y &\geq 6 \\ 3x + y &\leq 9 \\ x &\leq 2 \\ y &\leq 6 \\ x, y &\geq 0 \end{aligned}$$

   (i) Sketch the feasible region for this problem.

   [4 marks]

   (ii) If $Z_1 = x - y$ and $Z_2 = -4x - y$, state the values of x, y, Z_1, Z_2 at each vertex of the feasible region. Identify which of these vertices are inferior solutions and hence find the Non-Inferior Set (NIS).

   [4 marks]

   (iii) Let $Z(w) = (1 - w)Z_1 + wZ_2$, where $0 \leq w \leq 1$. Determine, as a function of w, the set of points of the feasible region for which $Z(w)$ is minimised.

   [5 marks]

   (iv) Suppose the goals for Z_1 and Z_2 are $G_1 = 2$ and $G_2 = 3$ respectively. Formulate a goal program in which the penalties for exceeding goals G_1 and G_2 are 2 and 1 per unit, respectively, and where there is no penalty for undershooting.

   [4 marks]

5. (a) Consider a single-item static continuous review model in which the set-up cost is K per order, the demand rate is D units per unit time and the holding cost is h per unit per unit time. If the order size is y units, then the Total Cost per Unit time is given by

$$TCU(y) = \frac{KD}{y} + \frac{hy}{2}.$$

Given that the Economic Order Quantity y^* is given by $y^* = \sqrt{2KD/h}$, find an expression for $TCU(y^*)$.

Calculate the values of y^* and $TCU(y^*)$ if $K = £50$ per order, $D = 8$ units per day and $h = £0.10$ per unit per day.

Show that, in general,

$$\frac{TCU(y)}{TCU(y^*)} = \frac{1}{2}\left(\left(\frac{y}{y^*}\right) + \left(\frac{y^*}{y}\right)\right)$$

Given the numerical values of K, D, h above, for what range of y values will $TCU(y)$ be within 15% of its minimal value $TCU(y^*)$?

[12 marks]

(b) For a quasi-static single-item continuous review model, demand is assumed to follow a Normal distribution with mean 20 units per week and standard deviation 2 units per day. If the delivery lead time is 1 week, determine the mean demand and standard deviation of the demand during the lead time. Hence find the level of buffer stock which should be kept to ensure that the probability of a stock-out does not exceed 5%.

(You may assume that if Z has a Normal distribution with mean 0 and variance 1 then the probability that Z exceeds 1.64 is approximately 0.05.)

[8 marks]

6. (a) Sketch the state transition diagram for an $(M/M/1)$ queueing system in which arrivals occur at mean rate λ and the server operates at mean rate μ.

Derive expressions, in terms of $\rho = \lambda/\mu$, for the following quantities, assuming in each case that $\rho < 1$ and that the system is in a steady state.

(i) The probability p_0 of there being no users in the system;

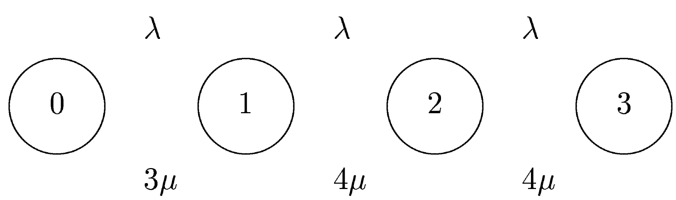
(ii) The probability p_n of there being n users in the system $(n = 1, 2, \ldots)$;

(ii) The expected number L_S of users in the system.

$\Big[$You may assume the formulae $\sum_{n=0}^{\infty} x^n = 1/(1-x)$, $\sum_{n=0}^{\infty} n x^{n-1} = 1/(1-x)^2$, $0 \le x < 1.\Big]$

[10 marks]

(b) A machine repair shop has room for no more than 3 machines at any one time. The average arrival rate of machines in need of repair is λ. Mechanics work together as a team repairing one machine at a time. There are normally 3 mechanics with a combined average repair rate of 3μ; however, a fourth mechanic joins the team whenever there are two or more machines in need of repair. Consequently the state transition diagram is as shown below.



If $\lambda = 5$ per week and $\mu = 2$ per week, then calculate the following quantities, assuming that the system is in a steady state.

(i) The probabilities p_0, p_1, p_2, p_3, where p_n is the probability that there are n machines under repair;

(ii) The average number of machines under repair L_S;

(iii) The *effective* arrival rate of machines at the repair shop λ_{eff}.

[10 marks]

7. (a) Solve the following transportation problem starting from the initial basic feasible solution given.

	A	B	C	D
J	[5] 10	[7] 8	[12]	[6]
K	[9]	[10]	[6] 3	[4] 4
L	[14]	[3] 2	[12]	[12] 13

What is the cost of (i) the given initial solution; (ii) your optimal solution?

[14 marks]

(b) Explain how the unbalanced problem below may be modelled as a *balanced* transportation problem and use the North West Corner Rule to provide an initial basic feasible solution. Assuming that there is zero cost associated with any surplus supply, is your initial solution optimal?

	A	B	C	Supply
U	[2]	[17]	[11]	15
V	[5]	[5]	[8]	15
Demand	8	10	6	

[6 marks]

/End