THE UNIVERSITY *of* LIVERPOOL

1. (a) A company manufactures two types of computer, type A and type B. Manufacturing a type A computer requires 2 hours of labour and 3 computer chips. Each type B computer requires 3 hours of labour and 1 computer chip. There are 900 hours of labour and 450 chips available per month. The company is able to sell up to 200 type A computers and up to 150 type B computers per month. For each type A computer sold the company makes a profit of £300, while the profit on each type B computer is £150, and the company's aim is to maximise monthly profits. Formulate this problem as a linear program. Sketch the feasible region of the problem, and find the optimal solution. Which, if any, of the constraints are redundant?

[14 marks]

(b) By making suitable variable change(s) and introducing extra variable(s) as appropriate, re-formulate the following linear program

P: minimise $y_0 = 4y_1 + 7y_2$ subject to

$$\begin{aligned} y_1 - 3y_2 &\leq 8 \\ 2y_1 + 5y_2 &\geq 9 \\ y_1, y_2 &\geq 0 \end{aligned}$$

in the following general canonical form:

maximise $x_0 = c_1x_1 + \cdots + c_nx_n$ subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} &= b_m \\ x_1, x_2, \ldots, x_{n+m} &\geq 0 \end{aligned}$$

[6 marks]

2. (a) Use the primal simplex method to solve the linear program

maximise $x_0 = 2x_1 + x_2 + 3x_3$
subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 18 \\ x_1 + x_2 + 3x_3 &\leq 24 \\ x_1 + 2x_2 + x_3 &\leq 16 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

[10 marks]

(b) Consider the linear program

maximise $x_0 = 3x_1 + ax_2$
subject to

$$\begin{aligned} x_1 + 4x_2 &\leq b \\ 2x_1 + x_2 &\leq 5 \end{aligned}$$

If $a = 1$ and $b = 10$ then introducing slack variables s_1, s_2 and applying the simplex method, one iteration yields the final optimal tableau below.

	x_1	x_2	s_1	s_2	
x_0	0	1/2	0	3/2	15/2
s_1	0	7/2	1	−1/2	15/2
x_1	1	1/2	0	1/2	5/2

Use this tableau to answer the following questions.

(i) State the optimal value of the objective function and the corresponding values of the decision variables.

(ii) With $a = 1$, how far can b decrease from its value of 10 without altering the optimal solution?

(iii) With $b = 10$, how far can a increase from its value of 1 without altering the optimal solution?

(iv) Find the optimal solution if $b = 10$ and the value of a is increased to $a = 4$.

[10 marks]

3. (a) Use the dual simplex method to solve the following linear program. (Solution by any other method will **not** receive credit.)

maximise $x_0 = -x_1 - x_2 - 3x_3$
subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\geq 9 \\ x_1 - x_2 + 2x_3 &\geq 6 \\ x_1 + 3x_2 - x_3 &\geq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

[14 marks]

(b) One method of solving the following linear program is to introduce artificial variables and apply the 'big M' method. Proceed with this method as far as writing down the initial tableau. (Note that you are **not** asked to perform any simplex iterations.)

maximise $x_0 = -4x_1 - 3x_2 + 4x_3$
subject to

$$\begin{aligned} x_1 + x_2 &\geq 4 \\ 2x_1 + 5x_3 &= 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

[6 marks]

4. Consider the bicriterion problem
minimise $\{Z_1, Z_2\}$
subject to

$$\begin{aligned} x + y &\geq 3 \\ 2x + y &\leq 8 \\ 2x + 3y &\leq 12 \\ x, y &\geq 0 \end{aligned}$$

(i) Sketch the feasible region for this problem.

[4 marks]

(ii) If $Z_1 = 2x - y$ and $Z_2 = x + 2y$, state the values of x, y, Z_1, Z_2 at each vertex of the feasible region. Identify which of these vertices are inferior solutions and hence find the Non-Inferior Set (NIS).

[4 marks]

(iii) Sketch the feasible region for this problem in **objective space**.

[3 marks]

(iv) Let $Z(w) = (1 - w)Z_1 + wZ_2$, where $0 \leq w \leq 1$. Determine, as a function of w, the set of points of the feasible region for which $Z(w)$ is minimised.

[5 marks]

(v) Find the Non-Inferior Set for the problem with the extra condition that x and y must be integers.

[4 marks]

5. (a) Consider a single-item static continuous review model in which the set-up cost is K per order, the demand rate is D units per unit time and the holding cost is h per unit per unit time. If the order size is y units, then derive an expression for the Total Cost per Unit time.

Hence show that the Economic Order Quantity is given by $y^* = \sqrt{2KD/h}$, and find an expression for the Total Cost per Unit time when the order size is y^*.

Calculate the value of y^* if $K = £100$ per order, $D = 12$ units per day and $h = £0.20$ per unit per day. If orders must be in multiples of 25 units, determine the order size which would lead to minimal cost.

Suppose now that replenishment of stock is not instantaneous, but takes place at rate r units per unit time. Given that Total Cost per Unit time is now

$$\text{TCU}(y) = \frac{KD}{y} + \frac{hy(1 - D/r)}{2}$$

write down an expression for the economic order quantity y^* in this case.

[12 marks]

(b) Describe briefly the main features of the *Just in Time* manufacturing system, outlining the main advantages of this system.

[8 marks]

6. (a) Sketch the state transition diagram for an $(M/M/2)$ queueing system in which arrivals occur at mean rate λ and each of the two servers operates at rate μ.

Defining $\rho = \lambda/\mu$ and assuming that $\rho < 2$, prove that the steady state probability p_0 of there being no users in the system is given by

$$p_0 = \frac{2-\rho}{2+\rho}$$

$\left[\text{You may assume the formula } \sum_{n=0}^{\infty} x^n = 1/(1-x)\ ,\ 0 \le x < 1.\right]$

[8 marks]

(b) A household consists of three people who share a single telephone. Each person arrives at the telephone at rate $\lambda = 0.4$ (time being measured in units of hours); if the phone is not in use, they make their call; if the phone is in use, they wait at the phone until it becomes available and then make their call. Each call lasts for an exponentially distributed time with mean $1/\mu = 0.2$ hours. The 'state of the system', n, is the number of people at the phone, including both those waiting and anyone making a call.

(i) Sketch the state transition diagram for this process.

(ii) Calculate the steady-state probabilities p_0, p_1, p_2, p_3, where p_n is the probability that the system is in state n.

(iii) Calculate the average number of people L_s at the phone.

(iv) Find the mean amount of time for which the phone is in use over a 16 hour day. (You may assume that the system is in steady-state throughout the day.)

[12 marks]

7. (a) Sources A, B, C produce candyfloss in amounts of 20, 42, 18 tons per month, respectively. The demands for candyfloss at destinations P, Q, R are 39, 34, 7 tons per month respectively. Transport costs (in £ per ton) are

		Destination		
		P	Q	R
	A	70	40	90
Source	B	80	120	50
	C	30	110	70

For the transportation problem of minimising total transport cost, use the North West Corner Rule to find an initial basic feasible solution. Is this solution optimal?

[6 marks]

(b) Solve the following transportation problem starting from the initial basic feasible solution given.

	P	Q	R	S
A	[9]	11 [5]	[0]	[2]
B	8 [3]	[10]	[6]	7 [7]
C	5 [6]	4 [3]	2 [20]	[12]

What is the cost of (i) the given initial solution; (ii) your optimal solution?

[14 marks]