1. Prove that the conic with equation

$$3x^2 - 2xy + 3y^2 + 8x - 8y - 8 = 0$$

is an ellipse. Find its centre and the equations of its axes.

Determine the canonical form of its equation, the lengths of its axes, its eccentricity and the distance of the foci from the centre.

Sketch the ellipse on graph paper after drawing precisely its axes and marking precisely the foci and the points of intersection of the ellipse with its axes. [Use the landscape position with a scale of 1 unit = 2 cm. Choose the origin near the centre of the graph paper and the coordinate axes parallel to the edges of the paper.] [25 marks]

2. A parametric curve is defined by

$$\mathbf{r}(t) = (t^2 - t^3, t^5) \quad (t \in \mathbf{R}).$$

Show that the curve has exactly one non-regular point and that this is a cusp. Determine the cuspidal tangent line, and prove that the cusp has order 3.

Determine the point P on the curve, not at the cusp, at which the tangent is parallel to the y-axis.

Show that the curve has exactly one linear point Q and that this point is a simple inflexion.

Plot and draw the curve on graph paper for $-1.1 \leq t \leq 1.1$ using a scale of 1 unit = 5 cm. Plot about 13 points including those given by $t = 0$ and $t = 1$. Indicate the points P and Q and the cuspidal tangent line. [Use the landscape position with the x-axis in the centre of the paper and the y-axis 2 cm. from the left hand side.] [25 marks]

3. Show that the cardioid

$$\mathbf{r}(t) = (2\cos t + \cos 2t, 2\sin t + \sin 2t) \quad (0 \le t \le 2\pi)$$

has speed

$$|\mathbf{r}'(t)| = 4\left|\cos\frac{t}{2}\right|.$$

[You may find the identity $\cos t = 2\cos^2\frac{t}{2} - 1$ useful.]

Deduce, or prove otherwise, that $\mathbf{r}$ has precisely one non-regular point at $t = \pi$. Show that this point is a cusp with horizontal tangent line. Show also that the tangent to the curve is parallel to the x-axis at the points given by $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

Show that, for $0 \le t_0 \le \pi$, the arc-length of the curve between the point given by $t = 0$ and that given by $t = t_0$ is $8\sin\frac{t_0}{2}$.

Plot and draw the curve for $0 \le t \le 2\pi$, paying special attention to points near $t = \pi$. Regarding t as an angle, plot points at intervals of 20°. Mark on your sketch the points corresponding to $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$. [Use the scale 1 unit = 2 cm. with the graph paper (ordinary, not polar) in the landscape position and the origin near the centre of the paper.] [25 marks]

4. Show that the parametric curve

$$\mathbf{r}(t) = (\cos t + t\sin t, \sin t - t\cos t) \quad (t \in \mathbf{R})$$

has exactly one non-regular point at $t = 0$.

Show that, for $t \neq 0$, the curvature κ of this curve is given by

$$\kappa(t) = \frac{1}{|t|}.$$

Hence calculate the limiting curvature and the limiting radius of curvature at the cusp. Deduce also that $\mathbf{r}$ has no linear point.

Show that the evolute of $\mathbf{r}$ has parametric equation

$$\mathbf{r}_*(t) = (\cos t, \sin t).$$

Plot and draw the original curve and the evolute on the same diagram for $-\pi \le t \le \pi$. Mark on your sketch the points on both curves corresponding to the values of the parameter $t = 0, \pm\frac{\pi}{2}, \pm\pi$. [Use the scale 1 unit = 2 cm. with the graph paper in the portrait position and the origin near the centre of the paper.] [25 marks]

5. Show that the limaçon

$$z(t) = (5 + \cos t)e^{it} \quad (0 \le t \le 2\pi)$$

is regular.

Using the formula

$$\kappa = -\frac{\mathrm{im}(z'\,\overline{z''})}{|z'|^3},$$

or otherwise, show that the curvature κ of this limaçon is given by

$$\kappa(t) = \frac{27 + 15\cos t}{(26 + 10\cos t)^{3/2}}.$$

Obtain also a formula for $\kappa'(t)$.

Show that the limaçon has no inflexions and precisely four vertices.

Plot the curve on polar graph paper (supplied) using a scale of 1 unit = 1 cm. and plotting points at intervals of 20°. Mark carefully on your sketch the the positions of the vertices. [25 marks]

6. Verify that, for any $\lambda \in \mathbf{R}$, the line given parametrically by

$$\mathbf{r}_\lambda(t) = \left(\frac{1}{2}\lambda - 8t, 4 + \lambda t\right) \quad (t \in \mathbf{R})$$

passes through the point $(\frac{1}{2}\lambda, 4)$ and is perpendicular to the line joining this point to the origin.

Determine the singular set of the family $\{\mathbf{r}_\lambda\}$ and show that the family has an envelope of the form

$$\mathbf{e}(u) = \left(u, 4 - \frac{1}{16}u^2\right).$$

Show that $\mathbf{e}$ is regular.

Using a scale of 1 unit = 1 cm., draw accurately on graph paper at least 18 of the lines $\mathbf{r}_\lambda$ for values of λ in the range $-10 \le \lambda \le 10$. [Use the landscape position with the origin near the centre of the paper.] [25 marks]

7. Consider the projective curve Π given by

$$y^2(z-x) - x^2(z+x) = 0$$

in homogeneous coordinates $(x:y:z)$. Show that Π has a unique singular point at $(0:0:1)$.

Show further that Π meets the line $z=0$ in a unique point $P=(0:1:0)$ and that the tangent line to Π at P is $x-z=0$.

The affine part Γ of Π, with $z=0$ as the line at infinity, is given by the equation

$$y^2(1-x) - x^2(1+x) = 0.$$

Show that Γ has a unique asymptote and find its equation. Show further that the origin $(x,y)=(0,0)$ is a node of Γ and find the tangent lines at the node.

Using the lines $y=tx$ through the origin, prove that the curve has a parametrisation

$$\mathbf{r}(t) = (x,y) = \left(\frac{t^2-1}{1+t^2}, \frac{t(t^2-1)}{1+t^2}\right).$$

Using the parametrisation, plot the curve on graph paper in the range $-5 \leq t \leq 5$ after first drawing the cuspidal tangent line and the asymptote. [Use the portrait position and a scale of 1 unit = 2 cm. with the origin near the centre of the paper.] [25 marks]