1. Prove that the conic with equation

$$x^2 + 6xy - 7y^2 + 2x - 26y - 23 = 0$$

is a hyperbola. Find its centre and the equations of its axes and its asymptotes.

Determine the canonical form of its equation, the length of its (transverse) semi-axis, its eccentricity and the distance of the foci from the centre.

Sketch the hyperbola on graph paper after drawing precisely its axes and marking precisely the foci and the points of intersection of the hyperbola with its transverse axis. [Use the landscape position with a scale of 1 unit = 1 cm. and choose the centre of the hyperbola near the centre of the graph paper and the coordinate axes parallel to the edges of the paper.] [25 marks]

2. A parametric curve is defined by

$$\mathbf{r}(t) = \left(\frac{1}{3}t^3 - \frac{1}{2}t^2, \frac{1}{4}t^4\right) \quad (t \in \mathbf{R}).$$

Show that the curve has exactly one non-regular point and that this is a cusp. Determine the cuspidal tangent line, and prove that the cusp has order 2.

Determine the point P on the curve, not at the cusp, at which the tangent is parallel to the y-axis.

Show that the curve has exactly one linear point Q and that this point is a simple inflexion.

Plot and draw the curve on graph paper for $-2.2 \leq t \leq 2.2$ using a scale of 1 unit = 2 cm. Plot about 13 points including those given by $t = 0$, $t = 1$ and $t = 2$. Indicate the points P and Q. [Use the landscape position with the x-axis 2 cm. from the bottom and the y-axis in the centre of the page.]

[25 marks]

3. The circle Γ, centre $(2, 0)$, radius 1, can be parametrised in the form

$$z(t) = 2 + e^{it} \quad (0 \leq t \leq 2\pi).$$

Show that the order of point-contact of Γ with the ellipse Γ_b defined by

$$\frac{x^2}{9} + \frac{y^2}{b^2} = 1,$$

where $b > 0$, at the point $z(0) = (3, 0)$ is at least 2 for any b. Find the unique value b_0 of b for which the order of point-contact > 2.

Calculate the radius of curvature of Γ_{b_0} at $(3, 0)$.

Now let $\mathbf{r}(u)$ be a parametrisation of Γ_{b_0} with constant speed v and suppose that $\mathbf{r}(u_0) = (3, 0)$. Find the length of the acceleration vector at u_0 (that is, $|\mathbf{r}''(u_0)|$) in terms of v.

Plot Γ and Γ_{b_0} on graph paper in the landscape position, taking a scale of 1 unit = 2 cm. [25 marks]

4. Show that the parabola

$$\mathbf{r}(t) = (t^2, 2t) \quad (t \in \mathbf{R})$$

is regular.

Show that the curvature κ of this curve is given by

$$\kappa(t) = -\frac{1}{2(1 + t^2)^{3/2}}.$$

Find a parametric equation for the evolute $\mathbf{r}_*(t)$, and verify that $\mathbf{r}_*$ has a non-regular point at $t = 0$.

Calculate the arc-length of the evolute between the point given by $t = 0$ and the point given by $t = 1$.

Plot and draw the original curve and the evolute on the same diagram for $-1.8 \leq t \leq 1.8$, paying special attention to points near $t = 0$. Mark on your sketch the points on both curves corresponding to the values of the parameter $t = 0, \pm 0.5, \pm 1, \pm 1.5, \pm 1.8$. [Use the scale 1 unit = 1 cm. with the graph paper in the portrait position, the y-axis on the extreme left hand side and the x-axis in the centre of the page.] [25 marks]

5. Show that the limaçon

$$z(t) = (a + b\cos t)e^{it} \quad (0 \le t \le 2\pi),$$

where a and b are positive real numbers with $a \ne b$, is regular.

Using the formula

$$\kappa = -\frac{im(z'\overline{z''})}{|z'|^3},$$

or otherwise, show that the curvature κ of the limaçon is given by

$$\kappa(t) = \frac{a^2 + 2b^2 + 3ab\cos t}{(a^2 + b^2 + 2ab\cos t)^{3/2}}.$$

Obtain also a formula for $\kappa'(t)$.

In the case where $a = 4$, $b = 2$, show that the limaçon has no simple inflexion, but does possess a point where $\kappa(t) = \kappa'(t) = 0$ (an undulation). Show further that the curve has precisely four vertices.

Sketch the curve on polar graph paper (supplied) using a scale of 1 unit = 1 cm. and plotting points at intervals of 20°. Mark carefully on your sketch the points of inflexion and the vertices. [25 marks]

6. Verify that, for any $\lambda \in \mathbf{R}$, the line given parametrically by

$$\mathbf{r}_\lambda(t) = (t\cos\lambda + (1-t)\cos 2\lambda, t\sin\lambda + (1-t)\sin 2\lambda) \quad (t \in \mathbf{R})$$

passes through the points $(\cos\lambda, \sin\lambda)$ and $(\cos 2\lambda, \sin 2\lambda)$ of the unit circle.

Determine the singular set of the family $\{\mathbf{r}_\lambda\}$ and show that the family has an envelope of the form

$$\mathbf{e}(u) = \left(\frac{2}{3}\cos u + \frac{1}{3}\cos 2u, \frac{2}{3}\sin u + \frac{1}{3}\sin 2u\right).$$

Show that $\mathbf{e}$ has precisely one non-regular point.

Using polar graph paper (supplied) and a scale of 1 unit = 5 cm., draw accurately at least 18 of the lines $\mathbf{r}_\lambda$. Mark the non-regular point of the envelope. [25 marks]

7. Consider the projective curve Π given by

$$x^3 + 3xy^2 + x^2z - y^2z = 0$$

in homogeneous coordinates $(x : y : z)$. Show that Π has a unique singular point at $(0 : 0 : 1)$.

Show further that Π meets the line $z = 0$ in a unique point $P = (0 : 1 : 0)$ and that the tangent line to Π at P is $3x - z = 0$.

The affine part Γ of Π, with $z = 0$ as the line at infinity, is given by the equation

$$x^3 + 3xy^2 + x^2 - y^2 = 0.$$

Show that Γ has a unique asymptote and find its equation. Show further that the origin $(x, y) = (0, 0)$ is a node of Γ and find the tangent lines at the node.

Using the lines $y = tx$ through the origin, prove that the curve has a parametrisation

$$\mathbf{r}(t) = (x, y) = \left(\frac{t^2 - 1}{1 + 3t^2}, \frac{t(t^2 - 1)}{1 + 3t^2} \right).$$

Using the parametrisation, plot the curve in the range $-6 \le t \le 6$ after first drawing the nodal tangent lines and the asymptote. [Use graph paper and a scale of 1 unit = 5 cm.] [25 marks]