1. Prove that the conic with equation

$$7x^2 - 8xy + y^2 - 20x + 14y - 23 = 0$$

is a hyperbola. Find its centre and the equations of its axes and its asymptotes.

Determine the canonical form of its equation, the length of its (transverse) semi-axis, its eccentricity and the distance of the foci from the centre.

Sketch the hyperbola on graph paper after drawing precisely its axes and asymptotes and marking precisely the foci and the points of intersection of the hyperbola with its transverse axis. [Use the portrait position with a scale of 1 unit = 1 cm. Choose the origin near the centre of the graph paper and the coordinate axes parallel to the edges of the paper.] [25 marks]

2. A parametric curve is defined by

$$\mathbf{r}(t) = (t^2, -t^4 + t^5) \quad (t \in \mathbf{R}).$$

Show that the curve has exactly one non-regular point and that this is a cusp of order 2. Determine the cuspidal tangent line.

Determine the point P on the curve, not at the cusp, at which the tangent is parallel to the x-axis.

Show that the curve has exactly one linear point Q and that this is a simple inflexion.

Plot and draw the curve on graph paper for $-0.8 \le t \le 1.3$ using a scale of 1 unit $= 10$ cm. Plot about 13 points including those given by $t = -0.8, 0, 0.8, 1, 1.3$. Mark on your sketch the points P and Q and the cuspidal tangent line. [Use the landscape position with the x-axis in the centre and the y-axis 1 cm. from the left hand side of the paper.] [25 marks]

3. Show that the cycloid

$$\mathbf{r}(t) = (t - \sin t, 1 - \cos t) \quad (t \in \mathbf{R})$$

has speed

$$|\mathbf{r}'(t)| = 2\left|\sin\frac{t}{2}\right|.$$

[You may find the identity $\cos 2\theta = 1 - 2\sin^2\theta$ useful.]

Deduce, or prove otherwise, that $\mathbf{r}$ has a non-regular point wherever $t = 2\pi n$ with n an integer. Show that each such point is a simple cusp with a vertical cuspidal tangent line.

Show also that the tangent to the curve is parallel to the x-axis whenever $t = (2n+1)\pi$ with n an integer.

Show that, for $2\pi \le t_0 \le 4\pi$, the arc-length of the curve between the point given by $t = 2\pi$ and that given by $t = t_0$ is $4(1 + \cos\frac{t_0}{2})$.

Plot and draw the curve for $0 \le t \le 4\pi$, paying special attention to points near $t = 2\pi$. Plot about 13 points including those given by $t = 0, \pi, 2\pi, 3\pi, 4\pi$. [Use the scale 1 unit = 2 cm. with the graph paper in the landscape position, the x-axis in the centre and the y-axis at the extreme left hand side of the paper.]

[25 marks]

4. Show that the parabola

$$\mathbf{r}(t) = (t^2, 2t) \quad (t \in \mathbf{R})$$

is regular.

Show that the curvature κ of this curve is given by

$$\kappa(t) = -\frac{1}{2(1+t^2)^{3/2}}.$$

Find a formula for $\kappa'(t)$ and show that the parabola has precisely one vertex.

Show that the evolute of $\mathbf{r}$ has parametric equation

$$\mathbf{r}_*(t) = (2 + 3t^2, -2t^3).$$

Verify that $\mathbf{r}_*$ is non-regular at the point corresponding to the vertex of $\mathbf{r}$.

Plot and draw the original curve and the evolute on the same diagram for $-1.8 \le t \le 1.8$, paying special attention to points near $t = 0$. Mark and label on your sketch the points on both curves corresponding to the values of the parameter $t = 0, \pm 0.5, \pm 1, \pm 1.5, \pm 1.8$. [Use the scale 1 unit = 1 cm. with the graph paper in the portrait position, the y-axis on the extreme left hand side and the x-axis in the centre of the paper.]

[25 marks]

5. Show that the limaçon

$$z(t) = (2 + \cos t)e^{it} \quad (0 \le t \le 2\pi)$$

is regular.

Using the formula

$$\kappa = -\frac{\mathrm{im}(z'\,\overline{z''})}{|z'|^3},$$

or otherwise, show that the curvature κ of this limaçon is given by

$$\kappa(t) = \frac{6 + 6\cos t}{(5 + 4\cos t)^{3/2}}.$$

Obtain also a formula for $\kappa'(t)$.

Show that the limaçon has no simple inflexion but does possess a point where $\kappa(t) = \kappa'(t)$ (an undulation). Show further that the curve has precisely four vertices (including the undulation).

Sketch the curve on polar graph paper (supplied) using a scale of 1 unit = 2 cm. and plotting points at intervals of 20°. Mark carefully on your sketch the undulation and the vertices. [25 marks]

6. Verify that, for any $\lambda \in \mathbf{R}$, the line given parametrically by

$$\mathbf{r}_\lambda(t) = (t\cos\lambda + (1-t)\cos 3\lambda, t\sin\lambda - (1-t)\sin 3\lambda) \quad (t \in \mathbf{R})$$

passes through the points $(\cos\lambda, \sin\lambda)$ and $(\cos 3\lambda, -\sin 3\lambda)$ of the unit circle.

Determine the singular set of the family $\{\mathbf{r}_\lambda\}$ and show that the family has an envelope of the form

$$\mathbf{e}(u) = \left(\frac{3}{2}\cos u - \frac{1}{2}\cos 3u, \frac{3}{2}\sin u + \frac{1}{2}\sin 3u\right).$$

Show that $\mathbf{e}$ has four non-regular points at $u = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. [You may find the formulae $\cos 3u = 4\cos^3 u - 3\cos u$, $\sin 3u = 3\sin u - 4\sin^3 u$ useful.]

Using polar graph paper (supplied) and a scale of 1 unit = 2 cm., draw accurately at least 18 of the lines $\mathbf{r}_\lambda$ for values of λ in the range $0 \le \lambda \le 2\pi$. Mark the non-regular points of the envelope on your sketch. [25 marks]

7. Consider the projective curve Π given by

$$x^4 + x^2yz - y^3z + y^4 = 0$$

in homogeneous coordinates $(x : y : z)$. Show that Π has a unique singular point at $(0 : 0 : 1)$.

Show further that Π does not meet the line $z = 0$.

The affine part Γ of Π, with $z = 0$ as the line at infinity, is given by the equation

$$x^4 + x^2y - y^3 + y^4 = 0.$$

Show that Γ is bounded. Show further that the origin $(x, y) = (0, 0)$ is a node of Γ of multiplicity 3 and find the tangent lines at the node.

Using the lines $y = tx$ through the origin, prove that the curve has a parametrisation

$$\mathbf{r}(t) = (x, y) = \left(\frac{t^3 - t}{1 + t^4}, \frac{t^4 - t^2}{1 + t^4}\right).$$

Show that the point $(0, 1)$ lies on Γ, but that there is no value of t for which $\mathbf{r}(t) = (0, 1)$.

Using the parametrisation, plot the curve on graph paper after first drawing the nodal tangent lines. [Use the landscape position and a scale of 1 unit = 10 cm. with the x-axis 3 cm. from the bottom of the paper. You should plot points for values of t in the ranges -5 to -1, -1 to 0, 0 to 1 and 1 to 5, and consider also what happens as $t \longrightarrow \pm\infty$.] [25 marks]