

1. Prove that the conic with equation

$$2x^2 - 4xy + 5y^2 + 4x - 16y - 22 = 0$$

is an ellipse. Find its centre and the equations of its axes.

Determine the canonical form of its equation, the lengths of its semi-axes, its eccentricity and the distance of the foci from the centre.

Sketch the ellipse on graph paper after drawing precisely its axes and marking precisely the foci and the points of intersection of the ellipse with its axes. [Use the landscape position with a scale of 1 unit = 1 cm. Choose the origin near the centre of the graph paper and the coordinate axes parallel to the edges of the paper.] [25 marks]

2. A parametric curve is defined by

$$\mathbf{r}(t) = (t^2, t^3 - t^4) \quad (t \in \mathbf{R}).$$

Show that the curve has exactly one non-regular point and that this is a simple cusp (i. e. a cusp of order 1). Determine the cuspidal tangent line.

Determine the point  $P$  on the curve, not at the cusp, at which the tangent is parallel to the  $x$ -axis.

Show that the curve has one linear point  $Q$  and that this point is a simple inflexion.

Plot and draw the curve on graph paper for  $-1.0 \leq t \leq 1.2$  using a scale of 1 unit = 10 cm. Plot about 13 points including that given by  $t = 0$ . Indicate the points  $P$  and  $Q$  and the cuspidal tangent line. [Use the portrait position with the  $x$ -axis 4 cm. from the top and the  $y$ -axis near the left hand side of the paper.] [25 marks]

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3. Show that the parametric curve

$$\mathbf{r}(t) = (8 \cos^3 t, 8 \sin^3 t) \quad (0 \leq t \leq 2\pi)$$

has speed

$$|\mathbf{r}'(t)| = 12|\sin 2t|.$$

Deduce, or prove otherwise, that  $\mathbf{r}$  has precisely four non-regular points at  $t = 0$ ,  $t = \frac{\pi}{2}$ ,  $t = \pi$  and  $t = \frac{3\pi}{2}$ . Show that each of these points is a cusp with either a horizontal or a vertical tangent line.

Show that, for  $0 \leq t_0 \leq \frac{\pi}{2}$ , the arc-length of the curve between the point given by  $t = 0$  and that given by  $t = t_0$  is  $6(1 - \cos 2t_0)$ .

Plot and draw the curve for  $0 \leq t \leq 2\pi$ , paying special attention to points near one of the cusps. Plot points at intervals of  $\frac{\pi}{8}$ . Mark on your sketch the points corresponding to the values of the parameter  $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$  and the cuspidal tangent line. [Use the scale 1 unit = 1 cm. with the graph paper in the portrait position and the origin near the centre of the paper.] [25 marks]

4. Show that the parametric curve

$$\mathbf{r}(t) = (\cosh t, \sinh t) \quad (t \in \mathbf{R})$$

is regular.

Show that the curvature  $\kappa$  of this curve is given by

$$\kappa(t) = -\frac{1}{(\sinh^2 t + \cosh^2 t)^{3/2}}$$

and that the evolute has parametric equation

$$\mathbf{r}_*(t) = (2 \cosh^3 t, -2 \sinh^3 t).$$

Verify that  $\mathbf{r}_*$  has a non-regular point at  $t = 0$ .

Plot and draw the original curve and the evolute on the same diagram for  $-1.2 \leq t \leq 1.2$ , paying special attention to points near  $t = 0$ . Mark on your sketch the points on both curves corresponding to the values of the parameter  $t = 0, \pm 0.8, \pm 1.2$ . [Use the scale 1 unit = 1 cm. with the graph paper in the portrait position, the  $x$ -axis in the centre and the  $y$ -axis near the left hand side of the paper.] [25 marks]

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5. Show that the limaçon

$$z(t) = (1 + 2 \cos t)e^{it} \quad (0 \leq t \leq 2\pi)$$

is regular, but possesses a multiple point corresponding to the values of the parameter  $t$  where  $1 + 2 \cos t = 0$ . Compute these values.

Using the formula

$$\kappa = -\frac{\operatorname{im}(z' \overline{z''})}{|z'|^3},$$

or otherwise, show that the curvature  $\kappa$  of the limaçon is given by

$$\kappa(t) = \frac{9 + 6 \cos t}{(5 + 4 \cos t)^{3/2}}.$$

Obtain also a formula for  $\kappa'(t)$ .

Show that the limaçon has no inflexions and precisely two vertices.

Plot the curve on polar graph paper (supplied) using a scale of 1 unit = 2 cm. and plotting points at intervals of  $20^\circ$ . Mark carefully on your sketch the multiple point and the vertices. [25 marks]

6. Verify that, for any  $\lambda \in \mathbf{R}$ , the line given parametrically by

$$\mathbf{r}_\lambda(t) = (2 \cos \lambda - 1 - 2t \sin \lambda, 2 \sin \lambda + t(2 \cos \lambda - 1)) \quad (t \in \mathbf{R})$$

passes through the point  $(2 \cos \lambda - 1, 2 \sin \lambda)$  and is perpendicular to the line joining this point to the origin.

Show that the singular set of the family  $\{\mathbf{r}_\lambda\}$  is given by

$$t = \frac{\sin \lambda}{2 - \cos \lambda}.$$

Deduce that the family has an envelope of the form

$$\mathbf{e}(u) = \frac{1}{2 - \cos u}(5 \cos u - 4, 3 \sin u).$$

Using a scale of 1 unit = 3.5 cm., draw accurately on graph paper at least 18 of the lines  $\mathbf{r}_\lambda$  for values of  $\lambda$  in the range  $0 \leq \lambda \leq 2\pi$ . [Use polar graph paper (supplied) in the landscape position and start by drawing a circle with centre at  $(-1, 0)$  (i.e. 3.5 cm. to the left of the central point) and radius 2 units; the points  $(2 \cos \lambda - 1, 2 \sin \lambda)$  then lie on this circle.] [25 marks]

7. Consider the projective curve  $\Pi$  given by

$$z(x^2 - y^2) - y^3 = 0$$

in homogeneous coordinates  $(x : y : z)$ . Show that  $\Pi$  has a unique singular point at  $(0 : 0 : 1)$ .

Show further that  $\Pi$  meets the line  $z = 0$  in a unique point  $P = (1 : 0 : 0)$  and that the tangent line to  $\Pi$  at  $P$  is  $z = 0$ .

The affine part  $\Gamma$  of  $\Pi$ , with  $z = 0$  as the line at infinity, is given by the equation

$$x^2 - y^2 - y^3 = 0.$$

Show that  $\Gamma$  has no asymptotes. Show further that the origin  $(x, y) = (0, 0)$  is a node of  $\Gamma$  and find the tangent lines at the node.

Using the lines  $x = ty$  through the origin, prove that the curve has a parametrisation

$$\mathbf{r}(t) = (x, y) = (t(t^2 - 1), t^2 - 1).$$

Using the parametrisation, plot the curve on graph paper in the range  $-2 \leq t \leq 2$  after first drawing the nodal tangent lines. [Use the landscape position and a scale of 1 unit = 2 cm. with the origin near the centre of the paper.]

[25 marks]