1. Prove that the conic with equation

$$3x^2 + 10xy + 3y^2 + 8x + 24y - 8 = 0$$

is a hyperbola. Find its centre and the equations of its axes and its asymptotes.

Determine the canonical form of its equation, the length of its (transverse) semi-axis, its eccentricity and the distance of the foci from the centre.

Sketch the hyperbola on graph paper after drawing precisely its axes and marking precisely the foci and the points of intersection of the hyperbola with its transverse axis. [Use the landscape position with a scale of 1 unit = 1 cm. and choose the centre of the hyperbola near the centre of the graph paper and the coordinate axes parallel to the edges of the paper.] [25 marks]

2. A parametric curve is defined by

$$\mathbf{r}(t) = (4t^2 - 4t^3, 2t^2 - 3t^4) \quad (t \in \mathbf{R}).$$

Show that the curve has exactly one non-regular point and that this is an ordinary cusp (i. e. a cusp of order 1). Determine the cuspidal tangent line.

Determine the point P on the curve, not at the cusp, at which the tangent is parallel to the y-axis.

Show that the curve has two linear points Q and R, both of which are simple inflexions.

Plot and draw the curve on graph paper for $-0.6 \le t \le 1$ using a scale of 1 unit = 10 cm. Plot about 13 points including those given by $t = 0$, $t = \frac{1}{3}$ and $t = 1$. Indicate the points P, Q and R and the cuspidal tangent line. [Use the landscape position with the x-axis 4 cm. from the top and the y-axis near the left hand side of the paper.] [25 marks]

3. Show that the cycloid

$$\mathbf{r}(t) = (2\pi t - \sin 2\pi t, 1 - \cos 2\pi t) \quad (t \in \mathbf{R})$$

has speed

$$|\mathbf{r}'(t)| = 4\pi|\sin \pi t|.$$

[You may find the identity $\cos 2\theta = 1 - 2\sin^2\theta$ useful.]

Deduce, or prove otherwise, that $\mathbf{r}$ has a non-regular point wherever the value of t is an integer. Show that each such point is a cusp with a vertical cuspidal tangent line.

Show also that the tangent to the curve is parallel to the x-axis whenever $t = n + \frac{1}{2}$ with n an integer.

Show that, for $0 \le t_0 \le 1$, the arc-length of the curve between the point given by $t = 0$ and that given by $t = t_0$ is $4(1 - \cos \pi t_0)$.

Plot and draw the curve for $-1 \le t \le 1$, paying special attention to points near $t = 0$. Plot about 13 points including those given by $t = 0$, $t = \pm 1$ and $t = \pm\frac{1}{2}$. [Use the scale 1 unit = 2 cm. with the graph paper in the landscape position and the origin near the centre of the paper.] [25 marks]

4. Show that the parametric curve

$$\mathbf{r}(t) = \left(t, \frac{1}{t}\right) \quad (t > 0)$$

is regular.

Show that the curvature κ of this curve is given by

$$\kappa(t) = \frac{2t^3}{(1+t^4)^{3/2}}$$

and that the evolute has parametric equation

$$\mathbf{r}_*(t) = \left(\frac{3t^4+1}{2t^3}, \frac{t^4+3}{2t}\right).$$

Verify that $\mathbf{r}_*$ has a non-regular point at $t = 1$.

Plot and draw the original curve and the evolute on the same diagram for $\frac{1}{3} \le t \le 3$, paying special attention to points near $t = 1$. Mark on your sketch the points on both curves corresponding to the values of the parameter $t = \frac{1}{3}, \frac{1}{2}, 1, 2, 3$. [Use the scale 1 unit = 1 cm. with the graph paper in the portrait position and the origin near the bottom left hand corner of the paper.]

[25 marks]

5. Show that the limaçon

$$z(t) = (1 + \cos t)e^{it} \quad (0 \leq t \leq 2\pi)$$

has precisely one non-regular point at $t = \pi$.

Using the formula

$$\kappa = -\frac{\operatorname{im}(z'\,\overline{z''})}{|z'|^3},$$

or otherwise, show that the curvature κ of this limaçon is given by

$$\kappa(t) = \frac{3}{2\sqrt{2(1 + \cos t)}}.$$

Obtain also a formula for $\kappa'(t)$.

Show that the limaçon has no inflexion and precisely one vertex.

Sketch the curve on polar graph paper (supplied) using a scale of 1 unit = 5 cm. and plotting points at intervals of 20°. Mark carefully on your sketch the non-regular point and the vertex. [25 marks]

6. Verify that, for any $\lambda \in \mathbf{R}$, the line given parametrically by

$$\mathbf{r}_\lambda(t) = (\lambda^2 - 2\lambda t, 2\lambda + (\lambda^2 - 1)t) \quad (t \in \mathbf{R})$$

passes through the point $(\lambda^2, 2\lambda)$ and is perpendicular to the line joining $(\lambda^2, 2\lambda)$ to $(1, 0)$.

Determine the singular set of the family $\{\mathbf{r}_\lambda\}$ and show that the family has an envelope of the form

$$\mathbf{e}(u) = (3u^2, 3u - u^3).$$

Show that $\mathbf{e}$ is regular but has one multiple point. Determine the coordinates of the multiple point and the two values of the parameter u corresponding to it.

Using a scale of 1 unit = 1 cm., draw accurately on graph paper at least 18 of the lines $\mathbf{r}_\lambda$ for values of λ in the range $-2 \leq \lambda \leq 2$. [Use the landscape position with the origin about 2 cm. to the left of the centre of the paper.] [25 marks]

7. Consider the projective curve Π given by

$$y(x^2 + y^2) - x^2 z = 0$$

in homogeneous coordinates $(x : y : z)$. Show that Π has a unique singular point at $(0 : 0 : 1)$.

Show further that Π meets the line $z = 0$ in a unique point $P = (1 : 0 : 0)$ and that the tangent line to Π at P is $y - z = 0$.

The affine part Γ of Π, with $z = 0$ as the line at infinity, is given by the equation

$$y(x^2 + y^2) - x^2 = 0.$$

Show that Γ has a unique asymptote and find its equation. Show further that the origin $(x, y) = (0, 0)$ is an ordinary cusp of Γ and find the cuspidal tangent line.

Using the lines $x = ty$ through the origin, prove that the curve has a parametrisation

$$\mathbf{r}(t) = (x, y) = \left(\frac{t^3}{1 + t^2}, \frac{t^2}{1 + t^2} \right).$$

Using the parametrisation, plot the curve on graph paper in the range $-3 \leq t \leq 3$ after first drawing the cuspidal tangent line and the asymptote. [Use the landscape position and a scale of 1 unit = 5 cm. with the origin near the centre of the paper.] [25 marks]