M244 2005 Solutions

Section A

1. To say that $\{v_1, v_2, \ldots, v_n\}$ *spans* V means that every element in V can be written as a linear combination

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n.$$

[1 mark]

Now, to show that W is a subspace of V, first note that the zero vector (0,0,0) is in W because the sum of its coordinates is 0+0+0=0. If now (x_1, y_1, z_1) and (x_2, y_2, z_2) are in W, then by definition $x_1 + y_1 + z_1 = 0$ and also $x_2 + y_2 + z_2 = 0$. So since

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2),$$

we add the three coordinates of this vector to obtain

$$(x_1 + x_2)+, (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0 + 0 = 0$$

Finally, if (x, y, z) is in W (so that $x + y + z + 0$) and λ is any real number, then $\lambda(x, y, z) = (\lambda x, \lambda y, \lambda z)$. Since

$$\lambda x + \lambda y, +\lambda z = \lambda(x + y + z) = \lambda \cdot 0 = 0$$

We have therefore shown that W is a subspace of V. [3 marks]

When we take the vectors $(1, 0, -1), (1, 2, 1)$ and $(2, -2, -4)$, it is clear that the first two are independent, so we investigate what happens if we write

$$(2, -2, -4) = \lambda(1, 0, -1) + \mu(1, 2, 1).$$

This leads to three equations: $2 = \lambda + \mu$, $-2 = 2\mu$ and $-4 = -\lambda + \mu$. We see that $\mu = -1$ and so $\lambda = 3$. Since these equations have non-zero solutions, the third vector depends on the first two so U has basis $(1, 0, -1)$ and $(1, 2, 1)$ and dimension 2. [2 marks]

Now

$$\begin{aligned} W &= \{(x, y, z) : x + y + z = 0\} \\ &= \{(x, y, z) : z = -y - x\} \\ &= \{(x, y, -y - x)\} \\ &= \{x(1, 0, -1) + y(0, 1, -1)\} \end{aligned}$$

Since $(1, 0, -1)$ and $(0, 1, 1)$ are clearly linearly independent, they are a basis for W so W also has dimension 2. [2 marks]

Now if (x, y, z) is in $U \cap W$, then $z = -y - x$ and so $(x, y, -y - x)$ is a linear combination of $(1, 0, -1)$ and $((1, 2, 1)$:

$$(x, y, -y - x) = \lambda(1, 0, -1) + \mu(1, 2, 1)$$

this gives $x = \lambda + \mu$, $y = 2\mu$ and $-y - x = -\lambda + \mu$. Thus $\mu = y/2$ and $\lambda = x - y/2$ (from the first two equations). The third then gives

$$-y - x = -\lambda + \mu = -x + y/2 + y/2 = -x + y$$

it follows that $y = 0$, so vectors of the form $(x, 0, -x)$ are in $U \cap V$ showing that this space has dimension 1. Since $U \cap V \neq \{0\}$, it follows that $\mathbf{R}^3$ is not the direct sum of U and V. [2 marks]

2. A group is a set G with a law of composition satisfying the following axioms:

(G1) for any $x, y \in G$, xy is in G;

(G2) for any x, y, z in G, $x(yz) = (xy)z$;

(G3) there is an element e in G such that for all $g \in G$, $ge = g = eg$;

(G4) given an element $g \in G$, there is an element g^{-1} of G with $gg^{-1} = 1 = g^{-1}g$.

Given two groups $(G, \circ)$ and $(H, \star)$, a map f is a homomorphism if

$$f(g \circ h) = f(g) \star f(h)$$

for all elements g, h of G

The kernel of f is the set of elements g in G such that $f(g) = e_H$.

The image of f is the set of those elements in h which are images of elements of G under f. [6 marks]

To show that ϕ is a homomorphism consider two matrices A, B in G, then

$$\phi(AB) = \phi\left(\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}\right) = \phi\left(\begin{pmatrix} a_1a_2 & 0 \\ 0 & b_1b_2 \end{pmatrix}\right) = a_1a_2.$$

Since $\phi(A) = a_1$ and $\phi(B) = a_2$ and the group operation in H is multiplication, we see that ϕ is a homomorphism.

[2 marks]

The kernel of ϕ is the set of matrices in G with '$a = 1$', so

$$\ker\,\phi = \{\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} : b \in \mathbf{R} \setminus \{0\}\}.$$

The image of ϕ is the whole of H since any non-zero real number could occur as the appropriate entry of an element A of G. [2 marks]

3. To show that L is linear, we check

$$L(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}) = L(\begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix})$$

$$= \begin{pmatrix} a_1 + a_2 + d_1 + d_2 & b_1 + b_2 \\ c_1 + c_2 & a_1 + a_2 + d_1 + d_2 \end{pmatrix}.$$

On the other hand

$$\begin{aligned} & L(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) + L(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}) \\ = & \begin{pmatrix} a_1 + d_1 & b_1 \\ c_1 & a_1 + d_1 \end{pmatrix} + \begin{pmatrix} a_2 + d_2 & b_2 \\ c_2 & a_2 + d_2 \end{pmatrix} \\ = & \begin{pmatrix} a_1 + a_2 + d_1 + d_2 & b_1 + b_2 \\ c_1 + c_2 & a_1 + a_2 + d_1 + d_2 \end{pmatrix} \end{aligned}$$

so LM1 holds. Similar checks hold for LM2:

$$\lambda L(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \lambda \begin{pmatrix} a + d & b \\ c & a + d \end{pmatrix} = \begin{pmatrix} \lambda(a + d) & \lambda b \\ \lambda c & \lambda(a + d) \end{pmatrix} = L(\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}).$$

[5 marks]

To compute the rank and nullity of L it is perhaps easiest to work out the matrix of L with respect to the standard basis. This is

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

If now $a + d = 0 = b = c$, we see that $a = -d$ so the kernel has dimension 1 (nullity 1). The image of L is spanned by the columns of M so is spanned by $(1, 0, 0, 1), (0, 1, 0, 0)$ and $(0, 0, 1, 0)$, so has dimension 3 (rank 3).

[4 marks]

4. We are given that $f((x_1, x_2), (y_1, y_2)) = x_1y_1 + 2x_1y_2 + x_2y_2$. Thus

$$\begin{aligned} f((2,2),(2,2)) &= 2\cdot 2 + 2\cdot 2\cdot 2 + 2\cdot 2 = 16 \\ f((2,2),(0,1)) &= 2\cdot 0 + 2\cdot 2\cdot 1 + 2\cdot 1 = 6 \\ f((0,1),(2,2)) &= 0\cdot 2 + 2\cdot 0\cdot 2 + 1\cdot 2 = 2 \\ f((0,1),(0,1)) &= 0\cdot 0 + 2\cdot 0\cdot 1 + 1\cdot 1 = 1 \end{aligned}$$

so the required matrix is $A = \begin{pmatrix} 16 & 6 \\ 2 & 1 \end{pmatrix}$ [3 marks]

Similarly for the basis (1,1), (0, 1)

$$\begin{aligned} f((1,1),(1,1)) &= 1\cdot 1 + 2\cdot 1\cdot 1 + 1\cdot 1 = 4 \\ f((1,1),(0,1)) &= 1\cdot 0 + 2\cdot 1\cdot 1 + 1\cdot 1 = 3 \\ f((0,1),(1,1)) &= 0\cdot 1 + 2\cdot 0\cdot 1 + 1\cdot 1 = 1 \\ f((0,1),(0,1)) &= 0\cdot 0 + 2\cdot 0\cdot 1 + 1\cdot 1 = 1 \end{aligned}$$

so, in this case, the required matrix is $B = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$. [3 marks]

Also $P = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$ so,

$$\begin{aligned} P^TAP &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 16 & 6 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 8 & 6 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \\ &= B \end{aligned}$$

as required. [3 marks]

5. The eigenvalues of A are the roots of the characteristic polynomial:

$$\begin{aligned}\det(\lambda I - A) &= \det\begin{pmatrix}\lambda-1 & -1 & -1\\ 0 & \lambda-2 & 4\\ 0 & 0 & \lambda-1\end{pmatrix}\\ &= (\lambda-1)((\lambda-2)(\lambda-1)+0)-(-1)\cdot 0+(-1)\cdot 0\\ &= (\lambda-1)^2(\lambda-2).\end{aligned}$$

[3 marks]

It follows that the eigenvalues are $\lambda = 2$ and $\lambda = 1$ (twice).
The eigenvectors when $\lambda = 2$ are given by $Au = 2u$ so that

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 4\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}2x\\ 2y\\ 2z\end{pmatrix}.$$

This leads to three equations:

$$x+y+z=2x; \quad 2y+4z=2y; \quad z=2z.$$

It follows that $z = 0$, and $y = x$ so a typical eigenvector is $v_1 = \begin{pmatrix}x\\ x\\ 0\end{pmatrix}$.

[2 marks]

The eigenvectors when $\lambda = 1$ are given by $Au = 1u$ so that

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 4\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}x\\ y\\ z\end{pmatrix}.$$

This leads to the equations $x+y+z = x$, $2y+4z = y$ and $z = z$. Thus $y+z = 0$ and $y = -4z$ so $y = z = 0$. If follows that a typical eigenvector for $\lambda = 1$ is $(x, 0, 0)$

[2 marks]

We now see that any set of three eigenvectors must contain at least two eigenvectors corresponding to the same eigenvector. Since the set of eigenvectors corresponding to the same eigenvalue is one diminsional, no set of three eigenvectors can be linearly independent.

[2 marks]

6. To show that v_1, v_2 and v_3 are a basis, it is sufficient to show that they are linearly independent, so we consider the equations

$$\lambda(1,1,1)+\mu(1,-1,2)+\nu(0,1,1)=(0,0,0).$$

We obtain three equations in the unknowns

$$\lambda + \mu = 0; \quad \lambda - \mu + \nu = 0; \quad \lambda + 2\mu + \nu = 0.$$

The first gives $\lambda = -\mu = 0$, so the others become $\nu = -2\lambda$ and $\nu = \lambda$. These clearly imply that $\lambda = \mu = \nu = 0$, so the given vectors are linearly independent and therefore form a basis.

[2 marks]

Now if ϕ_1, ϕ_2 and ϕ_3 are the dual basis for v_1, v_2 and v_3 we have

$$\begin{array}{lll} \phi_1(v_1) = 1; & \phi_1(v_2) = 0 & \phi_1(v_3) = 0 \\ \phi_1(v_2) = 0; & \phi_2(v_2) = 1 & \phi_2(v_3) = 0 \\ \phi_1(v_3) = 0; & \phi_3(v_2) = 0 & \phi_3(v_3) = 1 \end{array}$$

[1 marks]

Now if $\phi_1(x, y, z) = a_1x + b_1y + c_1z$, we obtain $a_1 + b_1 + c_1 = 1$, $a_1 - b_1 + 2c_1 = 0$ and $b_1 + c_1 = 0$. We now solve these equations for a_1, b_1, c_1 to get $b_1 = -c_1$ (so that $a_1 = 1$ and $1 - b_1 - 2b_1 = 0$). We can obtain $a_1 = 1$, $b_1 = 1/3$ and $c_1 = -1/3$, so that $\phi_1(x, y, x) = x + y/3 - z/3$.

Similar calculations are carried out to determine ϕ_2: we solve

$$a_2 + b_2 + c_2 = 0, \quad a_2 - b_2 + 2c_2 = 1, \text{ and } b_2 + c_2 = 0.$$

This time the solution is $a_2 = 0, b_2 = -1/3$ and $c_2 = 1/3$ so that ϕ_2 is given by $\phi_2(x, y, z) = -y/3 + z/3$.

For ϕ_3, we solve

$$a_3 + b_3 + c_3 = 0, \quad a_3 - b_3 + 2c_3 = 0 \text{ and } b_3 + c_3 = 1$$

These give $a_3 = -1, b_3 = 1/3$ and $c_3 = 2/3$ so that $\phi_3(x, y, x) = -x + y/3 + 2z/3$.

[5 marks]

Section B

7. The matrices A, B are respectively

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[2 marks]

The eigenvalues of A are the roots of $\det(\lambda I - A)$ or

$$\begin{aligned} & \det \begin{pmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & 0 \\ -1 & 0 & 0 & \lambda \end{pmatrix} \\ = \; & \lambda \det \begin{pmatrix} \lambda & -1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \det \begin{pmatrix} 0 & \lambda & -1 \\ 0 & -1 & \lambda \\ -1 & 0 & 0 \end{pmatrix} \\ = \; & \lambda^2 \det \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} - 1 \det \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} \\ = \; & \lambda^2(\lambda^2 - 1) - (\lambda^2 - 1) = (\lambda^2 - 1)^2 = (\lambda - 1)^2(\lambda + 1)^2 \end{aligned}$$

Thus A has eigenvalues 1(repeated twice) and -1 (repeated twice).

[5 marks]

When $\lambda = -1$ the eigenvectors are given by

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \\ -t \end{pmatrix}$$

so $x = -t$ and $y = -z$, so an eigenvector is $r + sx^2 - sx^3 - r$. Two obvious choices give us $1 - x^3$ and $x - x^2$ as eigenvalues. [2 marks]

When $\lambda = 1$ an eigenvector is given by

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

so we see $x = t$ and $y = z$ (no information), thus a typical eigenvector is $a(1+x^3)+b(x+x^2)$, giving $1+x^3$ and $x+x^2$ as obvious choices. [2 marks]

Thus the diagonal form of A is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[2 marks]

We now see that the second basis in the first part of the question, is actually a basis of eigenvectors, so when we change to this basis, we obtain D which is why $B = D$.

[2 marks].

8. The rank of f is the dimension of im f and the nullity of f is the dimension of its kernel. [2 marks]

The matrix of the given linear map is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To find a basis for the image of f, we need to find a basis for the space spanned by the columns of A: the vectors

$$(1,1,2,0); \quad (1,1,0,0); \quad (1,2,1,0); \text{ and } (1,1,2,0)$$

Clearly the last equals the first, so the only question is whether the third is a linear combination of the first 2. Consider

$$(1,2,1,0) = \lambda(1,1,2,0) + \mu(1,1,0,0)$$

This gives $1 = \lambda + \mu$, $2 = \lambda + \mu$ and $0 = 0$. Clearly the first two are inconsistent, so they have no solution, so the third vector is not a linear combination of the first two. We deduce that $(1,1,2,0)$, $(1,1,0,0)$ and $(1,2,1,0)$ are a basis for the image and so the rank of f is 3. [5 marks]

The kernel is the solution set for the equations $Au = 0$, giving

$$x+y+z+t = 0; \quad x+y+2z+t = 0; \quad 2x+z+2t = 0.$$

It is clear that if we subtract the first two equations, we obtain $z = 0$. Rewriting then gives $x+y+t = 0$ (twice) and $2x+2t = 0$. Thus $t = -x$,

$t = -2x$ and $y = 0$ so the solution set consists of vectors of the form $(x, 0, 0, -x) = x(1, 0, 0, -1)$. This is clearly a one dimensional space spanned by the vector $(1, 0, 0, 1)$ so the nullity is 1.

[5 marks]

To decide whether $\mathbf{R}^4$ is a direct sum of the kernel and the image of f or not, we try to find a u with $f(u) = 0$ and $u = f(v)$. Thus u of the form $x(1, 0, 0, -1)$ and also u is in the image of f so

$$u = \lambda(1, 1, 2, 0) + \mu(1, 1, 0, 0) + \nu(1, 2, 1, 0)$$

Since all vectors in the image of f have zero fourth coordinate,the only vector common to ker f and im f is that with $x = 0$ so the intersection of ker f and im f is $\{0\}$. Thus the sum of ker f and im f has dimension 4 and so must equal $\mathbf{R}^4$. It follows that $\mathbf{R}^4$ is the direct sum of ker f and im f.

[3 marks]

9. The given form is $q(x, y, z) = x^2 + 6xz - 2y^2 + z^2$ so its matrix is

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

[1 mark]

The eigenvalues of A are the zeros of the polynomial

$$\begin{aligned}
&= \det \begin{pmatrix} \lambda - 1 & -0 & -3 \\ 0 & \lambda + 2 & 0 \\ -3 & 0 & \lambda - 1 \end{pmatrix} \\
&= (\lambda - 1)\det \begin{pmatrix} \lambda + 2 & 0 \\ 0 & \lambda - 1 \end{pmatrix} - 3 \det \begin{pmatrix} 0 & \lambda + 2 \\ -3 & 0 \end{pmatrix} \\
&= (\lambda - 1)(\lambda + 2)(\lambda - 1) - 3(3(\lambda + 2)) \\
&= (\lambda + 2)(\lambda - 1)^2 - 9\lambda - 18 \\
&= (\lambda + 2)((\lambda - 1)^2 - 9) \\
&= (\lambda + 2)(\lambda^2 - 2\lambda - 8) \\
&= (\lambda + 2)(\lambda - 4)(\lambda + 2).
\end{aligned}$$

It follows that the eigenvalues are -2 (twice) and 4. [3 marks]

The eigenvectors for eigenvalue -2 are given by

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}$$

so we obtain the equations $x + 3z = -2x$ (or $3x + 3z = 0$), $-2y = -2y$ (so y is unconstrained) and $3x + z = -2z$ (also giving $x + y = 0$). Thus a typical eigenvector is $(x, y, -x)$. [3 marks]

The eigenvectors for eigenvalue 4 are given by

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}.$$

This time the equations are $x + 3z = 4x$ (or $x = z$), $-2y = 4y$ (giving $y = 0$) and $3x + z = 4z$ (so $x = z$). A typical eigenvector is of the form $(x, 0, x)$. [3 marks]

The required orthogonal matrix P is obtained by putting orthgonormal eigenvectors into columns so

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \text{ and } D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

[3 marks]

The surface becomes $4X^2 - 2Y^2 - 2Z^2 = 25$, a hyperboloid of two sheets with circular cross-section. [2 marks]

10 (i) To show e is unique, suppose that G had two identities e_1 and e_2 then $e_1 = g = ge_1$ and $e_2g = g = ge_2$ for all g in G. Now consider the element e_1e_2. Since e_1 is a left identity, this is e_2, and since e_2 is a right identity this is e_1 so $e_1 = e_2$. [2 marks]

(ii) Suppose that $a \circ b = g = a \circ c$ for some elements a, b, c in G. Multiply the equation $a \circ b = a \circ c$ on both sides by the inverse of a to get $a^{-1} \circ (a \circ b) = a^{-1} \circ (a \circ c)$. Now use associativity to get $(a^{-1} \circ a) \circ b = (a^{-1} \circ a) \circ c$. Since a^{-1} is the inverse for a, $a^{-1} \circ a = e$, so we obtain $e \circ b = e \circ c$. The result now follows since e is an identity element. [2 marks]

Now if an element g is repeated in the same row of a table, then g will be of the form $a \circ b$ and also of the form $a \circ c$ for some a, b, and c, so the above argument shows that $b = c$. [1 mark]

For columns, if $a \circ b = c \circ b$, we multiply on right by b^{-1} and again use associativity, inverse and identity to deduce that $a = c$. [2 marks]

(iii) Inspecting the given partial table, we see that $fa = a$ which can only happen in a group when f is the idenity element. This also means that b is the inverse of c (and so c is the inverse of b). Similarly, since d is the inverse of a, a is the inverse of d. We can now fill in more of the partial table:

$\circ$	a	b	c	d	f
a	b	c	?	f	a
b			f	a	b
c		f			c
d	f				d
f	a	b	c	d	f

The entry marked ? cannot be a, b, c or f (already in row) so must be d. Next consider the second entry in the column headed by b. This cannot be c, f or b (all in this column) or a, b, or f (already in row). This entry must also equal d. This gives

$\circ$	a	b	c	d	f
a	b	c	d	f	a
b		d	f	a	b
c		f			c
d	f				d
f	a	b	c	d	f

The remaining entry in the second row must now be c, that in the first column second row must then be d and the missing entry in the second column must be a. We now have

$\circ$	a	b	c	d	f
a	b	c	d	f	a
b	c	d	f	a	b
c	d	f		?	c
d	f	a			d
f	a	b	c	d	f

The entry at ? cannot be a, f or d (already in column) nor d, f, or c (already in row), so must be b the missing entry is that row is then a, that from the same column is c and the final entry b to complete the table as

$\circ$	a	b	c	d	f
a	b	c	d	f	a
b	c	d	f	a	b
c	d	f	a	b	c
d	f	a	b	c	d
f	a	b	c	d	f

[8 marks]