Math244 May 2005

SECTION A

1. Say what it means for $\{v_1, \ldots, v_k\}$ to *span* a vector space V.

Let U be the subspace of $\mathbf{R}^3$ spanned by $u_1 = (1, 0, -1), u_2 = (1, 2, 1)$ and $u_3 = (2, -2, -4)$. Let W be the set of vectors (x, y, z) in $\mathbf{R}^3$ where $x + y + z = 0$. Show that W is a subspace of $\mathbf{R}^3$. Calculate the dimensions of U and of W. Find the subspace $U \cap W$ and determine its dimension. Decide whether or not $\mathbf{R}^3 = U \oplus W$. Justify your answer. [10 marks]

2. Define the terms: *group, homomorphism, kernel, image.*

Consider the set G of 2×2 matrices of the form

$$G = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbf{R} \setminus \{0\}\},$$

under the operation of matrix multiplication. Let H be the group of non-zero real numbers, under the operation of multiplication [you need not show that these are groups]. Let $\phi : G \to H$ be defined by

$$\phi\Big(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\Big) = a.$$

Show that ϕ is a homomorphism. Find the kernel and image of ϕ. [10 marks]

3. Let V be the vector space of 2×2 real matrices. Let the map $L : V \to V$ be defined by

$$L\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big) = \begin{pmatrix} a+d & c \\ b & a+d \end{pmatrix}.$$

Prove that L is a linear map and compute its rank and nullity. [9 marks]

4. Let f be the bilinear form on $\mathbf{R}^2$ defined by

$$f((x_1, x_2), (y_1, y_2)) = x_1y_1 + 2x_1y_2 + x_2y_2.$$

Let $u_1 = (2, 2), u_2 = (0, 1)$. Compute $f(u_1, u_1), f(u_1, u_2), f(u_2, u_1), f(u_2, u_2)$. Find the matrix A of f relative to the basis $\{u_1, u_2\}$. Find the matrix B of f relative to the basis $\{v_1, v_2\}$, where $v_1 = (1, 1), v_2 = (0, 1)$.

Find the change of basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ and show that $B = P^T AP$. [9 marks]

5. Let A be the 3×3 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the eigenvalues and corresponding eigenvectors for A. Deduce that it is not possible to find three eigenvectors for A which are linearly independent.

[9 marks]

6. Consider the following three vectors in the space $V = \mathbf{R}^3$,

$$v_1 = (1, 1, 1), \quad v_2 = (1, -1, 2) \quad \text{and} \quad v_3 = (0, 1, 1).$$

Show that v_1, v_2 and v_3 form a basis for V. Let ϕ_1, ϕ_2 and ϕ_3 be the dual basis for $\{v_1, v_2, v_3\}$. Find an expression for the value of each of the maps ϕ_1, ϕ_2, ϕ_3 at a general point (x, y, z) of V. [8 marks]

SECTION B

7. Let V be the vector space of polynomials of degree at most 3 with real coefficients. Let $f : V \to V$ be the linear map defined by

$$f(a + bx + cx^2 + dx^3) = d + cx + bx^2 + ax^3.$$

Find A, the matrix of f with respect to the standard basis, $\{1, x, x^2, x^3\}$, of V and the matrix B of f with respect to the basis $\{1 + x^3, x + x^2, 1 - x^3, x - x^2\}$ of V.

Find the eigenvalues and corresponding eigenvectors of A and hence write down a diagonal matrix D equivalent to A. Explain the connection between B and D. [15 marks]

8. Define the *rank* and the *nullity* of a linear map.
Let $f : \mathbf{R}^4 \to \mathbf{R}^4$ be given by

$$f(x, y, z, t) = (x + y + z + t, x + y + 2z + t, 2x + z + 2t, 0).$$

Find a basis for U, the image of f and a basis for W, the kernel of f. Hence compute the rank of f and the nullity of f. Decide whether or not $\mathbf{R}^4$ is $U \oplus W$.

[15 marks]

9. Consider the quadratic form

$$q(x, y, z) = x^2 + 6xz - 2y^2 + z^2.$$

Write down the matrix A representing q with respect to the standard basis. Find a diagonal matrix D equivalent to A and an orthogonal matrix P which describes the change of basis from the standard basis to a basis in which q is diagonal. Describe geometrically the surface $q(x, y, z) = 25$. Draw a sketch of the surface.

[15 marks]

10. (i) Let G be a group. Show that the identity element e is unique.

(ii) Let G be a group with operation $*$ and let a, b, c be elements of G. Show that $a * b = a * c$ implies that $b = c$. Deduce that no element can be repeated in the same row inside a group table. Similarly show that no element can be repeated in the same column of the table.

(iii) The following is a partially completed group table for a group with five elements. Fill in the missing entries. You must justify (entry by entry) why each choice of entry is the only one possible.

$\circ$	a	b	c	d	f
a	b	c			
b			f	a	
c				b	
d	f				
f	a				

[15 marks]