THE UNIVERSITY
of LIVERPOOL

SOLUTIONS FOR MATH244 (MAY 2007)

SECTION A

1. We compute:

$$\begin{aligned} f(u_1,u_1) &= -2\cdot 1\cdot 1 + 1\cdot 3 - 3\cdot 1 + 3\cdot 3 = 7,\\ f(u_1,u_2) &= 13,\\ f(u_2,u_1) &= 3,\\ f(u_2,u_2) &= 2. \end{aligned}$$

So, the matrix of f wrt u_1, u_2 is

$$A = \begin{pmatrix} 7 & 13 \\ 3 & 2 \end{pmatrix}.$$

[3 marks]

Similarly,the matrix of f wrt v_1, v_2 is $B = \begin{pmatrix} -7 & 3 \\ 13 & -2 \end{pmatrix}$.

[3 marks]

To compute the change-of-basis matrix, we write v_j as linear combinations of the u_j. (This will involve solving a system of linear equations.)

$$\begin{aligned} (-2,-1) &= -1\cdot(1,3) + 1\cdot(-1,2)\\ (3,4) &= 2\cdot(1,3) - 1\cdot(-1,2). \end{aligned}$$

So the change-of-basis matrix is $P = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$.

Alternatively, we can obtain P as the composition of change-of-basis matrices from the given bases to the standard basis.

$$P = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -2 & 3 \\ -1 & 4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ -1 & 4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -5 & 10 \\ 5 & -5 \end{pmatrix}.$$

Finally, it is easily checked that

$$P^T A P = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 7 & 13 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = B.$$

[3 marks].

9 marks in total for Question 1

Whole question: seen similar in exercises.

2.

(a) The vectors $v_1 \dots, v_k$ are a *basis* of the vector space V if they are linearly independent and span the vector space V. (Alternatively, every vector of V can be *uniquely* written as a linear combination of $v_1 \dots, v_k$.)

[2 marks]. *Standard definition from lectures.*

(b) The easiest way to solve this exercise is to put the vectors u_1, u_2, u_3 as the rows of a matrix, and use row operations to transform this matrix to reduced row echelon form. We obtain the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix},$$

which means that $(1, 0, 2)$ and $(0, 1, -3)$ form a basis of U, and hence the dimension of U is two.

(Of course there are plenty of other methods; for example, we might note that u_1 and u_2 are linearly independent, and that u_3 is a linear combination of the two. Any two linearly independent vectors in $U = \{(x, y, z) : z = 2x - 3y\}$ form a correct basis of U.)

[3 marks]. *Standard exercise.*

(c) Again, it is probably simplest to transform the basis into reduced row echelon form. We should obtain the same basis (recall that the reduced row echelon form is unique!), and hence $U = W$. Again, there are plenty of alternative methods to solve this question; e.g. by noticing that all three vectors w_1, w_2, w_3 belong to U. Since W clearly has dimension at least two (w_2 is not a scalar multiple of w_1), this also implies $U = W$. **[4 marks].** *Standard exercise.*

9 marks in total for Question 2

3.

(a) Let $e_1 = x^2$, $e_2 = x$, $e_3 = 1$. Then
$$\varphi(e_1) = -x^2 + 2x = -1 \cdot e_1 + 2 \cdot e_2 + 0 \cdot e_3,$$
so that the first column of the matrix should have entries $-1, 2, 0$. Proceeding similarly for e_2 and, we get
$$M = \begin{pmatrix} -1 & -2 & 4 \\ 2 & 4 & -8 \\ 0 & 1 & -3 \end{pmatrix}.$$

[3 marks] *Seen similar in exercises.*

(b) We now compute
$$\det(\lambda I - M) = \cdots = \lambda^3 - \lambda = \lambda(\lambda + 1)(\lambda - 1).$$
So the eigenvalues of λ are 0, 1 and -1.

[3 marks] *Standard exercise.*

To find the eigenvectors corresponding to these eigenvalues, we must solve the equations $Mv = 0$, $(I - M)v = 0$ and $(I + M)v = 0$. Doing this in the usual way, we obtain the eigenvectors $r \cdot (2, -3, -1)$, $r \cdot (-2, 4, 1)$ and $r \cdot (-1, 2, 1)$, respectively.

[3 marks] *Standard exercise.*

(c) In particular, the matrix M is diagonalizable, since we can find a basis of three linearly independent eigenvectors, e.g. $(2, -3, -1), (-2, 4, 1), (-1, 2, 1)$.

[2 marks] *Standard exercise.*

11 marks in total for Question 3

4. A *group* is a set G together with a binary operation $*$ such that: **(G1)** for all $g_1, g_2 \in G$, $g_1 * g_2 \in G$; **(G2)** for all $g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$; **(G3)** there exists an element $e \in G$ such that, for all $g \in G$, $e * g = g * e = g$; **(G4)** for every $g \in G$, there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.

[2 marks]. *Standard definition from lectures.*

If G, H are groups, then a map $\varphi : G \to H$ is a *homomorphism* if, for all $g_1, g_2 \in G$, $\varphi(g_1 *_1 g_2) = \varphi(g_1) *_2 \varphi(g_2)$, where $*_1$ is the group law in G and $*_2$ is the group law in H.

[1 marks]. *Standard definition from lectures.*

The map φ is *injective* if, for all $g_1, g_2 \in G$, $\varphi(g_1) = \varphi(g_2) \Rightarrow g_1 = g_2$. The map φ is *surjective* if, for all $h \in H$, there exists $g \in G$ such that $\varphi(g) = h$.

[2 marks]. *Standard definitions from lectures.*

Let s, t be arbitrary non-zero real numbers. We have

$$\varphi(s+t) = \begin{pmatrix} e^{s+t} & 0 \\ 0 & e^{-(s+t)} \end{pmatrix} = \begin{pmatrix} e^s \cdot e^t & 0 \\ 0 & e^{-s} \cdot e^{-t} \end{pmatrix} = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \varphi(s)\varphi(t).$$

Hence φ is a homomorphism.

[2 marks]. *Seen somewhat similar in exercises.*

If $\varphi(s) = \varphi(t)$, then (comparing the top left entries), we must have $e^s = e^t$, and hence (applying the natural logarithm to both sides) $s = t$, so φ is injective. The map φ is clearly not surjective, as e.g.

$$\varphi(t) \neq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for all $x \in G$.

[2 marks]. *Seen similar in exercises.*
9 marks in total for Question 4

5.

(a) A function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is an *isometry of the plane* if φ is bijective and preserves the Euclidean distance:

$$\|\varphi(v) - \varphi(w)\| = \|v - w\|$$

for all $v, w \in \mathbb{R}^2$.

[1 mark]. *Standard definition from lectures.*

(b) The composition $\varphi \circ \psi$ of two isometries is again an isometry:

$$\|\varphi(\psi(v)) - \varphi(\psi(w))\| = \|\psi(v) - \psi(w)\| = \|v - w\|.$$

[2 marks].

Associativity is clearly satisfied. The neutral element is given by the identity map $\varphi(v) = v$. This is an isometry because

$$\|\varphi(v) - \varphi(w)\| = \|v - w\|$$

by definition. **[2 marks]**

Since an isometry φ is bijective, it has an inverse $\psi = \varphi^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$. This inverse is also an isometry, because we have

$$\|\psi(v) - \psi(w)\| = \|\varphi(\psi(v)) - \varphi(\psi(w))\| = \|v - w\|,$$

where we first used the fact that φ is an isometry, and then the fact that φ is the inverse of ψ.

[3 marks]. *Similar examples seen in exercises and lecture.*

8 marks in total for Question 5

6. The *rank* of φ is the dimension of $\mathrm{Im}(\varphi)$. The *nullity* of φ is the dimension of $\ker(\varphi)$.

[1 mark]. *Standard definitions from lectures.*

The rank and nullity theorem states that

$$\dim V = \mathrm{rank}(\varphi) + \mathrm{nullity}(\varphi).$$

[1 mark]. *Standard theorem from lectures.*

The map φ from the question can be written as

$$\varphi(ax^3 + bx^2 + cx + d) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & -1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

and we know from lectures that such maps are linear. Alternatively, we can verify this directly by checking the definition of a linear map.

[2 marks]. *Standard exercise.*

We can find the kernel of φ quite easily, by taking an arbitrary polynomial $v = ax^3 + bx^2 + cx + d$ in V, and seeing when $\varphi(v) = 0$. This leads to a system of linear equations, which we can solve using the usual methods; we will see that the dimension of the kernel is 2. (More precisely, we have

$$\ker(\varphi) = \{ax^3 + bx^2 + cx + d : a + c - 2d = 0 \text{ and } b - c + 3d = 0\};$$

a possible basis hence might consist of $x^3 - bx^2 - cx$ and $2x^3 - 3x^2 + 1$. So $\mathrm{nullity}(\varphi) = 2$ and $\mathrm{rank}(\varphi) = 4 - 2 = 2$.

(Of course there are other possible ways of finding the rank and the nullity; for example, one might find a basis for the image of φ by considering the span of the image vectors $(1, 2, 3)$, $(1, 1, 2)$, $(0, 1, 1)$ and $(1, -1, 0)$ of the standard basis of V. This should lead to the observation that the image of φ is precisely the subspace spanned by $(1, 0, 1)$ and $(0, 1, 1)$; e.g., the set $\{(x, y, z) : z = x + y\}$.)

[4 marks]. *Standard exercise.*

Since $\mathrm{nullity}(\varphi) \neq 0$, φ is not an isomorphism.

[1 mark]. *Standard exercise.*

9 marks in total for Question 6

Section B

7. Let $V = \mathrm{Pol}_3(\mathbb{R})$ be the vector space of polynomials of degree at most three with real coefficients; let

$$U = \{ax^3 + bx^2 + cx + d : d + a = b + 2c\}$$

and

$$W = \{(2a + 2b)x^3 + 2ax^2 + (3a - b)x - 2b : a, b \in \mathbb{R}\}.$$

Then U is the kernel of the linear map

$$V \to \mathbb{R}; ax^3 + bx^2 + cx + d \mapsto (1, -1, -2, 1) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

(we know from lectures that maps of this form are linear), so U is a subspace. Alternatively, we can check the usual subspace criteria to see that U is a subspace.

[4 marks].

A basis of U is given by the three vectors $x^3 - 1$, $x^2 + 1$ and $x + 2$. (These vectors are clearly linearly independent, and every element of U can be written as a linear combination of these three vectors.) So $\dim(U) = 3$.

The subspace W is, by definition, the span of the two vectors $2x^3 + 2x^2 + 3x$ and $2x^3 - x - 2$; since these are linearly independent, they form a basis of W. So $\dim(W) = 2$.

[4 marks].

To find the intersection, we can use several methods. For example, we can let $v = (2a + 2b)x^3 + 2ax^2 + (3a - b)x - 2b$ be an arbitrary element of W, and see when $v \in U$; this is the case when $2a + 2b - 2b = 2a + 2(3a - b)$; i.e., when $b = 3a$. It follows that the vector $4x^3 + x^2 - 3$ forms a basis of $U \cap W$.

[4 marks].

We thus have $\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 3 + 2 - 1 = 4$.

[2 marks].

Since $U \cap W \neq \{0\}$, V is not the direct sum of U and W.

[1 mark].

15 marks in total for Question 7 *Seen similar in exercises*

8.

$$q(x, y, z) = -2x^2 + 4xy + 2y^2 - 2xz + 2z^2.$$

with respect to the standard bases is

$$A = \begin{pmatrix} -2 & 2 & -1 \\ 2 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

[3 marks].

We can find a basis with respect to which q is diagonal either by finding a basis consisting of orthogonal eigenvectors of A, or by using the "matrix method" from lectures. The latter method is easier, but will not necessarily yield an orthogonal basis. If you use the former method, you should get that the eigenvalues are $-3, 2, 3$ with corresponding eigenvectors $(5, -2, 1)$, $(0, 1, 2)$, $(1, 2, -1)$. These eigenvectors form the desired basis.

A matrix P with the desired property is thus given by

$$P = \begin{pmatrix} 5 & 0 & 1 \\ -2 & 1 & 2 \\ 1 & 2 & -1 \end{pmatrix}.$$

The desired diagonal matrix is

$$D = P^T AP = \begin{pmatrix} -90 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 18 \end{pmatrix}.$$

[9 marks].

The diagonal matrix has full rank, so the rank of q is 3. The signature is the number of positive entries minus the number of negative entries, and is thus 1. The surface is a hyperboloid of one sheet.

[3 marks].

15 marks in total for Question 8 *Seen similar in exercises.*

9.

(a) We have

$$\ker(\varphi) = \{(a,b,c,d) : a+b=c, a+b+d=0, b+c+3d=0, a+2b+3d=0\};$$

i.e., $\ker(\varphi)$ is the solution set of four linear equations in the four variables a,b,c,d. We can solve this system e.g. by reduction to row echelon form, and should obtain the equivalent system consisting of the three equations $a=d$, $b=-2d$ and $c=-d$. A basis for the solution space is then given e.g. by the single vector $(1,-2,-1,1)$.

[3 marks]. *Standard exercise.*

In particular, we see that $\operatorname{rank}(\varphi) = 4-1 = 3$, so we only need to find three linearly independent vectors in the image of φ. For example, it is easy to see that the images of the first three basis vectors, $v_1 = x^3+x^2+1$, $v_2 = x^3+x^2+x+2$ and $v_3 = x - x^3$ are linearly independent. (Indeed, v_1 and v_3 are linearly independent, and v_2 can clearly not be written as a linear combination of these two.) So they form a basis of the image of φ.

Alternatively, write the four vectors with respect to the standard basis of $\mathrm{Pol}_3(\mathbb{R})$, put them as the rows of a matrix and reduce to row echelon form. Any three linearly independent vectors from

$$\mathrm{Im}(\varphi) = \{ax^3+bx^2+cx+d : d = a+b\}$$

gives a correct answer. Only two marks will be given if the basis is given as vectors from $\mathbb{R}^4$, rather than as polynomials.

[4 marks]. *Standard exercise.*

(b) The change-of-basis-matrix is given by writing the four vectors in B with respect to the standard basis and putting the coefficients as the columns of P:

$$P = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}.$$

[2 marks]. *Standard exercise*

The inverse of P can be found by the usual method. ; it is

$$P^{-1} = \begin{pmatrix} -1 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}.$$

[4 marks]. *Seen similarly in exercises.*

(c) M can be found by writing down the matrix of φ with respect to the standard bases, and using the matrix P^{-1} from the previous part of the question:

$$M = P^{-1} \cdot \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Alternatively, we can find write the images of the standard vectors of $\mathbb{R}^4$ in terms of the basis B; the corresponding coefficients will form the columns of M.

[**2 marks**]. *Seen similarly in exercises.*

15 marks in total for Question 9

10.

(a) Statement (i) is false; any noncommutative group is a counterexample. For example, let G be the symmetry group of a triangle, let a be anticlockwise rotation and let b be a reflection.

[2 marks]. *Seen somewhat similarly in lectures.*

Statement (ii) is true (this follows by multiplying the equation by a^{-1} from the left).

[1 mark]. *Seen in Lectures.*

Statement (iii) is true; it is one of the group axioms.

[1 marks]. *Unseen.*

Statement (iv) is false. Consider, for example, $G = C_3$ be the cyclic group of order three, and let a be a generator of G. Then $a^3 = e$, but $a \neq e$.

[2 marks]. *Unseen.*

(b) Since $BC = B$, C must be the identity element of the group, and we can fill in the corresponding column and row.

[1 mark]. *Seen similarly in exercises.*

Since $AB = C$, B is the inverse of A, so we also have $BA = C$.

[1 marks]. *Seen similarly in exercises.*

We can use the fact that no element can be repeated in a row or column of a group table to fill in e.g. the row corresponding to E, and continue this way to fill in most of the table:

*	A	B	C	D	E	F
A	?	C	A	?	?	?
B	C	A	B	?	?	?
C	A	B	C	D	E	F
D	F	E	D	C	B	A
E	D	F	E	A	C	B
F	?	D	F	?	A	C.

[2 marks]. *Seen similarly in exercises.*

To fill in the final entries, we need to use the group laws. For example, you might notice that $AD = (ED)D = E(DD) = EC = E$. Now the rest of the table can be easily filled in.

*	A	B	C	D	E	F
A	B	C	A	E	F	D
B	C	A	B	F	D	E
C	A	B	C	D	E	F
D	F	E	D	C	B	A
E	D	F	E	A	C	B
F	E	D	F	B	A	C.

[3 marks]. *Unseen.*

(c) The group of symmetries of a triangle (or equivalently the symmetric group S_3 (or, equivalently, the dihedral group D_6) has the same group table.

[2 marks]. *Unseen.*

15 marks in total for Question 10