M244 2003 Solutions

Section A

1. To say that $\{v_1, v_2, \ldots, v_n\}$ *spans* V means that every element in V can be written as a linear combination

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n.$$

[1 mark]

When we take the vectors $(1,0,-1), (1,-2,1)$ and $(2,2,-4)$, it is clear that the first two are independent, so we investigate what happens if we write

$$(2,2,-4) = \lambda(1,0,-1) + \mu(1,-2,1).$$

This leads to three equations: $2 = \lambda + \mu$, $2 = -2\mu$ and $-4 = -\lambda + \mu$. We see that $\mu = -1$ and so $\lambda = 3$. Since these equations have non-zero solutions, the third vector depends on the first two so U has basis $(1,0,-1)$ and $(1,-2,1)$ and dimension 2. [2 marks]

To show that W is a subspace, note that the zero vector $(0,0,0)$ is in W because $0+0+0=0$. If (x_1,y_1,z_1) and (x_2,y_2,z_2) are in W, so that $x_1+y_1+z_1=0$ and $x_2+y_2+z_2=0$, then

$$(x_1,y_1,z_1)+(x_2,y_2,z_2) = (x_1+x_2, y_1+y_2, z_1+z_2)$$

and since

$$x_1+x_2+y_1+y_2+z_1+z_2 = (x_1+y_1+z_1)+(x_2+y_2+z_3) = 0+0=0,$$

it follows that $(x_1+x_2, y_1+y_2, z_1+z_2)$ is in W. Finally, let (x,y,z) be in W and λ be any real number. Then $x+y+z=0$ and $\lambda(x,y,z) = (\lambda x, \lambda y, \lambda z)$. Since

$$\lambda x + \lambda y + \lambda z = \lambda(x+y+z) = \lambda 0 = 0,$$

is follows that $\lambda(x,y,z)$ is also in W and that W is a subspace of V.

[3 marks]

Now

$$\begin{aligned} W &= \{(x,y,z) : x+y+z=0\} \\ &= \{(x,y,z) : z = -(x+y)\} \\ &= \{(x,y,-(x+y))\} \\ &= \{x(1,0,-1)+y(0,1,-1)\} \end{aligned}$$

Since $(1,0,-1)$ and $(0,1,-1)$ are clearly linearly indepedent, they are a basis for W so W also has dimension 2. [2 marks]

Now if (x,y,z) is in $U \cap W$, then $z = -(x+y)$ and so $(x,y,-(x+y))$ is a linear combination of $(1,0,-1)$ and $((1,-2,1)$:

$$(x,y,-(x+y)) = \lambda(1,0,-1) + \mu(1,2,1)$$

this gives $x = \lambda + \mu$, $y = 2\mu$ and $-(x+y) = -\lambda + \mu$. Thus $\mu = y/2$ and $\lambda = x - y/s$ (from the first two equations). The third is then also satisfied, so every element in W is an element of U (also because the 2 basis vectors are in W). It follows that $U = W$, so $U \cap W = U$ and $U + W = U$.

[2 marks]

2. A group is a set G with a law of composition satisfying the following axioms:

(G1) for any $x, y \in G$, xy is in G;

(G2) for any x, y, z in G, $x(yz) = (xy)z$;

(G3) there is an element e in G such that for all $g \in G$, $ge = g = eg$;

(G4) given an element $g \in G$, there is an element g^{-1} of G with $gg^{-1} = e = g^{-1}g$.

Given two groups $(G, \circ)$ and $(H, \star)$, a map f is a homomorphism if

$$f(g \circ h) = f(g) \star f(h)$$

for all elements g, h of G .

The kernel of f is the set of elements g in G such that $f(g) = e_H$.

The image of f is the set of those elements in h which are images of elements of G under f. [6 marks]

To show that ϕ is a homomorphism consider two matrices A, B in G, then

$$\begin{aligned}
\phi(AB) &= \phi\left(\begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix}\right) \\
&= \phi\left(\begin{pmatrix} 1 & a_1 + a_2 & b_1 + c_2 a_1 + b_2 \\ 0 & 1 & c_1 + c_2 \\ 0 & 0 & 1 \end{pmatrix}\right) \\
&= a_1 + a_2
\end{aligned}$$

Since $\phi(A) = a_1$ and $\phi(B) = a_2$ and the group operation in H is addition, we see that ϕ is a homomorphism. [2 marks]

The kernel of ϕ is the set of matrices in G with '$a = 0$', so

$$\ker\phi = \{\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : b, c \in \mathbf{R}\}.$$

The image of ϕ is the whole of $\mathbf{R}$ since any real number could occur as the appropriate entry of an element A of G. [2 marks]

3. Since $L(1) = x^3 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 1 \cdot x^3$, the entries in the first column of M are 0, 0, 0, 1. Similarly, we have $L(x) = x^2$, $L(x^2) = x$ and $L(x^3) = 1$. It follows that the matrix M is

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

[2 marks]

We next compute $\det(\lambda I - M)$ to get

$$\begin{aligned} \det(\lambda I - M) &= \\ &= \det \begin{pmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & 0 \\ -1 & 0 & 0 & \lambda \end{pmatrix} \\ &= \lambda \begin{pmatrix} \lambda & -1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - (-1) \begin{pmatrix} 0 & \lambda & -1 \\ 0 & -1 & \lambda \\ -1 & 0 & 0 \end{pmatrix} \\ &= \lambda(\lambda^3 - (-1)(-\lambda + 0)) + (-\lambda(\lambda) + 1) \\ &= \lambda^4 - \lambda^2 - \lambda^2 + 1 \\ &= (\lambda^2 - 1)^2 \end{aligned}$$

It follows that M has two repeated eigenvalues, namely 1 (twice) and -1 (twice). [4 marks]

When $\lambda = 1$, a vector $v = a + bx + cx^2 + dx^3$ is an eigenvector if $L(v) = v$, so $d + cx + bx^2 + ax^3 = a + bx + cx^2 + dx^3$. This occurs precisely if $d = a$ and $b = c$, so the eigenvectors are the polynomials of the form $a + bx + bx^2 + ax^3$.

[2 marks]

When $\lambda = -1$, a vector $v = a + bx + cx^2 + dx^3$ is an eigenvector if $L(v) = -v$, so $d + cx + bx^2 + ax^3 = -a - bx - cx^2 - dx^3$. This occurs precisely if $d = -a$ and $b = -c$, so the eigenvectors are the polynomials of the form $a + bx - bx^2 - ax^3$. [2 marks]

4. The map ϕ will take the vector $(1, 0)$ to that obtained by rotating anti-clockwise through 90° so (1,0) maps to (0,1) and $(0, 1)$ itself maps to $(-1, 0)$. Thus the matrix of ϕ is $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. [2 marks]

Since ℓ is the y-axis, $(1, 0)$ is mapped by σ_ℓ to $(-1, 0)$, and (0,1) is mapped to itself. It follows that A, the matrix representing σ_ℓ is $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. [2 marks]

Also k is the line $x = y$, so $(1, 0)$ is mapped by σ_ℓ to $(0, 1)$, and (0,1) is mapped to (1,0). It follows that B, the matrix representing σ_k is $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. [2 marks]

Finally the composite map will have matrix

$$AB = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

[2 marks]

This is the matrix M, which represents a rotation anti-clockwise through 90°. The powers of M are $M^2 = -I$, $M^3 = -M$ and $M^4 = I$, so the required integer is 4. This shows that after 4 rotation through 90°, one returns to the starting position. [2 marks]

5. We are given that $f((x_1, x_2), (y_1, y_2)) = x_1y_1 - x_1y_2 + x_2y_2$. Thus

$$\begin{aligned} f((2, 2), (2, 2)) &= 2 \cdot 2 - 2 \cdot 2 + 2 \cdot 2 = 4 \\ f((2, 2), (0, 1)) &= 2 \cdot 0 - 2 \cdot 1 + 2 \cdot 1 = 0 \\ f((0, 1), (2, 2)) &= 0 \cdot 2 - 0 \cdot 2 + 1 \cdot 2 = 2 \\ f((0, 1), (0, 1)) &= 0 \cdot 0 - 0 \cdot 1 + 1 \cdot 1 = 1 \end{aligned}$$

so the required matrix is $A = \begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix}$. [3 marks]

Similarly for the basis (1,1), (0, -1)

$$\begin{aligned}
f((1,1),(1,1)) &= 1\cdot 1 - 1\cdot 1 + 1\cdot 1 = 1\\
f((1,1),(0,-1)) &= 1\cdot 0 - 1\cdot(-1) + 1\cdot(-1) = 0\\
f((0,-1),(1,1)) &= 0\cdot 1 - 0\cdot 1 + (-1)\cdot 1 = -1\\
f((0,-1),(0,-1)) &= 0\cdot 0 - 0\cdot -1 + (-1)\cdot(-1) = 1 = 1
\end{aligned}$$

so, in this case, the required matrix is $B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. [3 marks]

Also $P = \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}$ so,

$$\begin{aligned}
P^T AP &= \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}\\
&= \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\\
&= B
\end{aligned}$$

as required. [3 marks]

6. The rank of f is the dimension of the image of f and the nullity is the dimension of the kernel of f.

[2 marks]

To find the kernel of f, find those (x, y, z) with $x+y-z = 0$, $x-y+2z = 0$ and $2x + z = 0$. Any method for doing this is accepted, but the last says $2x = -z$ and the first then says $3x + y = 0$. Since this is precisely the second equation the solution set is $(x, -3x, -2x)$ Thus ker$f$ has dimension 1 (spanned by (1, -3, -2)) and the nullity of f is 1.

[2 marks].

The image of f is the space spanned by the columns of the matrix of f, so is spanned by $(1, 1, 2), (1, -1, 0)$ and $(-1, 2, 1)$. Since the third is dependent on the first two ($2(-1, 2, 1) = (1, 1, 2) - 3(1, -1, 0)$) and the first two are (by inspection) linearly independent, this space has dimension 2 and the rank is 2.

[2 marks]

Section B

7. To show that U is a subspace note

the zero polynomial is in U (take $a = b = d = 0$);

if $a_1 + b_1x + (a_1 + b_1)x^2 + d_1x^3$ and $a_2 + b_2x + (a_2 + b_2)x^2 + d_2x^3$ are in U then

$$(a_1 + b_1x + (a_1 + b_1)x^2 + d_1x^3 + (a_2 + b_2x + (a_2 + b_2)x^2 + d_2x^3)$$

is equal to

$$(a_1 + a_2) + (b_1 + b_2)x + (a_1 + b_1 + a_2 + b_2)x^2 + (d_1 + d_2)x^3$$

Since this is in U, U is closed under addition;

finally,

$$\lambda(a + bx + (a + b)x^2 + dx^3) = \lambda a + \lambda bx + \lambda(a + b)x^2 + \lambda dx^3.$$

We have therefore shown that U is a subspace.

Similarly for W, we check the standard requirements:

the zero polynomial is in W (take $a = 0$);

if $a_1 + a_1x + a_1x^2 + a_1x^3$ and $a_2 + a_2x + a_2x^2 + a_2x^3$ are in W then their sum is $(a_1 + a_2) + (a_1 + a_2)x + (a_1 + a_2)x^2 + (a_1 + a_2)x^3$, which is also in W, so W is closed under addition;

finally,

$$\lambda(a + ax + ax^2 + a^3) = \lambda a + \lambda ax + \lambda ax^2 + \lambda ax^3.$$

We have therefore shown that W is a subspace. [4 marks]

To find the dimension of U, note that $1 + x^2, x + x^2$ and x^3 are all in U. These are clearly linearly independent: if

$$\lambda(1 + x^2) + \mu(x + x^2) + \nu x^3 = 0$$

then equating the constant terms gives $\lambda = 0$, equating coefficents of x gives $\mu = 0$ and equating coefficients of x^3 gives $\nu = 0$. They also span U because

$$a + bx + (a + b)x^2 + dx^3 = a(1 + x^2) + b(x + x^2) + dx^3.$$

These polynomials therefore form a basis and U has dimension 3. [3 marks]

Next, every element of W is of the form $a(1 + x + x^2 + x^3)$, so W has a basis of one element and so has dimension 1. [2 marks]

Next, if $f(x)$ is in $U \cap W$ then $f(x)$ has all coefficients equal (since it is in W) but the third coefficient is the sum of the first two (since F is in W). It follows that $U \cap W = \{0\}$ so this intersection has dimension zero.

[2 marks]

Finally, we see that 1 is in $U + W$ by taking

$$(-x - x^2 - x^3) + (1 + x + x^2 + x^3)$$

with $-x - x^2 - x^3$ being an element of U and $1 + x + x^2 + x^3$ being an element of W. Similarly,

$$\begin{aligned} x &= (-1 - x^2 - x^3) + (1 + x + x^2 + x^3) \text{ and} \\ x^2 &= (1 + x + 2x^2 + x^3) - (1 + x + x^2 + x^3) \end{aligned}$$

since x^3 is also in $U + W$ (because x^3 is in W). Thus each standard basis vector is in $U + W$, so $U + W = V$. [3 marks]

From the information we have already calculated, it is clear that V is the direct sum of U and W. [1 mark]

8. The dual space is defined to be the set of all linear maps from V to $\mathbf{R}$. Given θ, ϕ in V^*, we can define $\theta + \phi$ by $(\theta + \phi)(x) = \theta(x) + \phi(x)$. Similarly, for λ in $\mathbf{R}$, we define $(\lambda\theta)(x) = \lambda(\theta(x))$.

Given a basis $\{x_1, x_2, \ldots, x_n\}$ for V, we define ϕ_i as the unique linear map which maps x_i to 1, but all other basis elements to 0. To prove this gives a dual basis, suppose first that f is any linear map from V to $\mathbf{R}$. Let λ_j be that scalar which f maps x_j to (so that $\lambda_j = f(x_j)$). Then for any j the map $\lambda_1\phi_1 + \cdots \lambda_n\phi_n$ takes x_j to λ_j (since $\phi_i(x_j) = 0$ for $i \neq j$). Thus the maps f and $\lambda_1\phi_1 + \cdots \lambda_n\phi_n$ agree in their action on a basis for V so must be equal and the vectors $\phi_1, \ldots \phi_n$ span V^*. Now to check linear independence, suppose that $\lambda_1\phi_1 + \cdots \lambda_n\phi_n = 0$. Then, for any x_j, $(\lambda_1\phi_1 + \cdots \lambda_n\phi_n)(x_j) = 0$ we also know that $(\lambda_1\phi_1 + \cdots \lambda_n\phi_n)(x_j) = \lambda_j$, so each λ_j would then be zero. Thus $\{\phi_1, \ldots, \phi_n\}$ is a basis for V^*. [7 marks]

Thus we have that

$$\begin{array}{lll} \phi_1(v_1) = 1; & \phi_1(v_2) = 0 & \phi_1(v_3) = 0 \\ \phi_1(v_2) = 0; & \phi_2(v_2) = 1 & \phi_2(v_3) = 0 \\ \phi_1(v_3) = 0; & \phi_3(v_2) = 0 & \phi_3(v_3) = 1. \end{array}$$

[1 mark]

Now if $\phi_1(x, y, z) = a_1x + b_1y + c_1z$, we obtain $a_1 + b_1 + c_1 = 1$, $a_1 + 2b_1 + 4c_1 = 0$ and $a_1 - b_1 + c_1 = 0$. We now solve these equations for a_1, b_1, c_1 to get $2a_1 + 2c_1 = 1$ (so $c_1 = 1/2 - a_1$). We can now re-write the first two to say $b_1 + 1/2 = 1$ (so $b_1 = 1/2$) and $a_1 + 4c_1 = -1$ (so $a_1 = +1$ and $c_1 = -1/2$), so that $\phi_1(x, y, x) = x + y/2 - z/2$. Similar calculations are carried out to determine ϕ_2: we solve

$$a_2 + b_2 + c_2 = 0, \quad a_2 + 2b_2 + 4c_2 = 1, \text{ and } a_2 - b_2 + c_2 = 0$$

These give $a_2 = -1/3, b_2 = 0$ and $c_2 = 1/3$ so that $\phi_2(x, y, x) = -x/3 + z/3$. For ϕ_3, we solve

$$a_3 + b_3 + c_3 = 0, \quad a_3 + 2b_3 + 4c_3 = 0, \text{ and } a_3 - b_3 + c_3 = 1.$$

This time the solution is $a_3 = 1/3, b_3 = -1/2$ and $c_3 = 1/6$ so that ϕ_3 is given by $\phi_3(x, , y, z) = x/3 - y/2 + z/6$. [5 marks]

Finally

$$\begin{aligned} \phi_1(3,2,1) &= 3 + 2/2 - 1/2 = 7/2; \\ \phi_2(3,2,1) &= -3/3 + 1/3 = -2/3; \\ \phi_3(3,2,1) &= 3/3 - 2/2 + 1/6 = 1/6. \end{aligned}$$

[2 marks]

9. The given form is $q(x, y, z) = x^2 + 6xy + y^2 + 4z^2$ so its matrix is

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

[1 mark]

The eigenvalues of A are the zeros of the polynomial

$$\begin{aligned} \det(\lambda I - A) &= \\ \det \begin{pmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 4 \end{pmatrix} &= \\ (\lambda - 1)\det \begin{pmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 4 \end{pmatrix} + 3 \det \begin{pmatrix} -3 & 0 \\ 0 & \lambda - 4 \end{pmatrix} &= \\ (\lambda - 1)(\lambda - 1)(\lambda - 4) + 3(-3\lambda + 12) &= \\ (\lambda - 4)((\lambda - 1)^2 - 9) &= \\ (\lambda - 4)(\lambda^2 - 2\lambda - 8) &= \\ (\lambda - 4)(\lambda - 4)(\lambda + 2). & \end{aligned}$$

It follows that the eigenvalues are 4 (twice) and -2. [4 marks]

The eigenvectors for eigenvalue -2 are given by

$$\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}$$

so we obtain the equations $x + 3y = -2x$ (or $x + y = 0$), $3x + y = -2y$ (also giving $x + y = 0$) and $z = -2z$ (so $z = 0$). Thus a typical eigenvector is $(x, -x, 0)$. [2 marks]

The eigenvectors for eigenvalue 4 are given by

$$\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

This time the equations are $x + 3y = 4x$ (or $x = y$) $3x + y = 4y$ (also giving $x = y$) and $4z = 4z$ (so no constraints on z). A typical eigenvector is of the form $x(1, 1, 0) + z(0, 0, 1)$. [3 marks]

The required P is obtained by putting these eigenvectors into columns so

$$P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

[2 marks]

The surface becomes $-2X^2 + 4Y^2 + 4Z^2 = 25$, a hyperboloid of one sheet with circular cross-sections on planes parallel to the XY-plane (cooling tower shape) [3 marks]

10 (i) To show e is unique, suppose that G had two identities e_1 and e_2 then $e_1 = g = ge_1$ and $e_2 g = g = ge_2$ for all g in G. Now consider the element $e_1 e_2$. Since e_1 is a left identity, this is e_2, and since e_2 is a right identity this is e_1 so $e_1 = e_2$. [2 marks]

(ii) Suppose that $a \circ b = g = a \circ c$ for some elements a, b, c in G. Multiply the equation $a \circ b = a \circ c$ on both sides by the inverse of a to get $a^{-1} \circ (a \circ b) = a^{-1} \circ (a \circ c)$. Now use associativity to get $(a^{-1} \circ a) \circ b = (a^{-1} \circ a) \circ c$. Since a^{-1} is the inverse for a, $a^{-1} \circ a = e$, so we obtain $e \circ b = e \circ c$. The result now follows since e is an identity element. [2 marks]

Now if an element g is repeated in the same row of a table, then g will be of the form $a \circ b$ and also of the form $a \circ c$ for some a, b, and c, so the above argument shows that $b = c$. [1 mark]

For columns, if $a \circ b = c \circ b$, we multiply on right by b^{-1} and again use associativity, inverse and identity to deduce that $a = c$. [2 marks]

(iii) Inspecting the given table, we see that $b \circ (c \circ d) = b \circ a = d$, whereas $(b \circ c) \circ d = a \circ d = b$, so the operation is not associative. [3 marks]

Now suppose G is a group so that we have (from the given information) a partial table

	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c		?
b	b		e		
c	c			e	
d	d				e

If G is to be a group, the entry marked ? cannot be a, e or c (already in row or d (already in column), so must be b. This makes the other missing entry in this row d. Giving

	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b		e	?	
c	c			e	
d	d				e

The entry marked now is not c, d or b, e so must be a. This makes the last entry in this row c and the second entry d.

	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	d	e	a	c
c	c			e	?
d	d				e

The missing entry in the last column must be a, the third entry in that row must then be d and the second b. The final row then fills in uniquely and we obtain the table given at the start of the question which we have shown is non-associative, so G is not a group. [5 marks]