THE UNIVERSITY
of LIVERPOOL

SOLUTIONS FOR MATH244 (SEPTEMBER 2006)

SECTION A

1.

(a) The span of $\{v_1, \dots, v_k\}$ is the set of all linear combinations of $v_1, \dots, v_k$:

$$\operatorname{span}(v_1, \dots, v_k) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k : \lambda_1, \dots, \lambda_k \in \mathbb{K}\}.$$

(It is acceptable if students just cover the case of a real vector space, writing $\mathbb{R}$ instead of $\mathbb{K}$.) **[2 marks].** *Standard definition from lectures.*

(b) *First method*: First put u_1, u_2, u_3 as the rows of a matrix, and use row operations to reduce to echelon form. Solution:

$$\begin{pmatrix} 3 & 0 & 1 \\ 1 & -2 & 1 \\ 1 & 4 & -1 \end{pmatrix} \longrightarrow \dots \longrightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $(3, 0, 1), (0, 3, -1)$ is a basis of U, and the dimension is 2.

Second method: Find a nontrivial solution to the equation $\lambda u_1 + \mu u_2 + \nu u_3 = 0$; e.g. $(3, 0, 1) - 2(1, -2, 1) - (1, 4, -1) = (0, 0, 0)$. So the three vectors are linearly dependent, so $\dim U < 3$. On the other hand, there are clearly two linearly independent vectors among the three vectors given (any pair will do), so $\dim U \geq 2$.

Remark: An easy way to check whether a given basis for U is correct is to note that $U = \{(x, y, z) : x - y = 3z\}$.

[3 marks]. *Standard exercise.*

(c) *First method*: Again, put w_1, w_2, w_3 as the rows of a matrix, and use row operations to reduce to echelon form:

$$\begin{pmatrix} 2 & 2 & 0 \\ -4 & 2 & -2 \\ 5 & 2 & 1 \end{pmatrix} \longrightarrow \dots \longrightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the space W also has the basis $\{(3, 0, 1), (0, 3, -1)\}$, and so $U = W$.

Second method: Since we have already computed the dimension of U as 2, and the dimension of W is clearly at least 2, it is enough to check that $W \subset U$; i.e., each of the vectors w_j belongs to U. This can be done, for example, by writing them as linear combinations of u_1 and u_2 (again solving a system of linear equations):

$$w_1 = u_1 - u_2, \quad w_2 = -u_1 - u_2, \quad w_3 = 2u_1 - u_2.$$

[4 marks]. *Standard exercise.*

9 marks in total for Question 1

2. A *group* is a set G together with a binary operation $*$ such that: **(G1)** for all $g_1, g_2 \in G$, $g_1 * g_2 \in G$; **(G2)** for all $g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$; **(G3)** there exists an element $e \in G$ such that, for all $g \in G$, $e * g = g * e = g$; **(G4)** for every $g \in G$, there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.

[**2 marks**]. *Standard definition from lectures.*

If G, H are groups, then a map $\varphi : G \to H$ is a *homomorphism* if, for all $g_1, g_2 \in G$, $\varphi(g_1 *_1 g_2) = \varphi(g_1) *_2 \varphi(g_2)$, where $*_1$ is the group law in G and $*_2$ is the group law in H.

[**1 marks**]. *Standard definition from lectures.*

The map φ is *injective* if, for all $g_1, g_2 \in G$, $\varphi(g_1) = \varphi(g_2) \Rightarrow g_1 = g_2$. The map φ is *surjective* if, for all $h \in H$, there exists $g \in G$ such that $\varphi(g) = h$.

[**2 marks**]. *Standard definitions from lectures.*

Let x, y be arbitrary non-zero real numbers. We have

$$\varphi(xy) = \begin{pmatrix} xy & 0 \\ 0 & (xy)^2 \end{pmatrix} = \begin{pmatrix} xy & 0 \\ 0 & x^2y^2 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x^2 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^2 \end{pmatrix} = \varphi(x)\varphi(y).$$

Hence φ is a homomorphism.

[**2 marks**]. *Seen somewhat similar in exercises.*

If $\varphi(x) = \varphi(y)$, then (comparing the top left entries), we must have $x = y$, so φ is injective. The map φ is clearly not surjective, as e.g.

$$\varphi(x) \neq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for all $x \in G$.

[**2 marks**]. *Seen similar in exercises.*

9 marks in total for Question 2

3.

(a) A function $\varphi : G \to G$ is an *isomorphism* if φ is a homomorphism, injective and surjective.

[2 marks]. *Standard definition from lectures.*

(b) The composition $\varphi \circ \psi$ of two isomorphisms is again an isomorphism. Indeed, we see that the composition is still a homomorphism:

$$\varphi(\psi(v_1 v_2)) = \varphi(\psi(v_1)\psi(v_2)) = \varphi(\psi(v_1))\varphi(\psi(v_2)).$$

If $\varphi(\psi(v_1)) = \varphi(\psi(v_2))$, then $\psi(v_1) = \psi(v_2)$ by injectivity of φ, and thus $v_1 = v_2$ by injectivity of ψ. So $\varphi \circ \psi$ is injective.

Let $w \in V$. Then by surjectivity of φ, there is $v_1 \in V$ such that $\varphi(v_1) = w$. By surjectivity of ψ, there is $v \in V$ such that $\psi(v) = v_1$. Then $\varphi(\psi(v)) = \varphi(v_1) = w$, so $\varphi \circ \psi$ is surjective.

[4 marks].

Associativity is clearly satisfied. The neutral element is given by the identity map $\varphi(v) = v$. The inverse element of φ is given by its inverse φ^{-1}.

[3 marks]. *Similar examples seen in exercises and lecture.*

9 marks in total for Question 3

4.

(a) Let e_1, e_2, e_3, e_4 be the standard basis vectors of $\mathbb{R}^4$. Then

$$\varphi(e_1) = (-1, 0, 4, 0) = -1 \cdot e_1 + 4 \cdot e_3,$$

so that the first column of the matrix should have entries $-1, 0, 4, 0$. Proceeding similarly for e_2, e_3 and e_4, we get

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 3 & 4 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[3 marks] *Seen similar in exercises.*

(b) We now compute

$$\det(\lambda I - M) = \begin{vmatrix} (\lambda+1) & 0 & 0 & 0 \\ 0 & (\lambda-3) & 0 & 0 \\ -4 & 0 & (\lambda-3) & -4 \\ 0 & 0 & 0 & (\lambda+1) \end{vmatrix}$$

$$= (\lambda+1) \begin{vmatrix} (\lambda-3) & 0 & 0 \\ 0 & (\lambda-3) & -4 \\ 0 & 0 & (\lambda+1) \end{vmatrix} = (\lambda+1)^2(\lambda-3)^2.$$

So the eigenvalues of λ are -1 and 3.

[3 marks] *Standard exercise.*

To find the eigenvectors corresponding to these eigenvalues, we must solve the equations $(I + M)v = 0$ and $(3I - M)v = 0$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 4 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & -4 \\ 0 & 0 & 0 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So we see that the eigenvectors with eigenvalue -1 are of the form $(\lambda, 0, \mu, -\lambda - \mu)$ and those with eigenvalue 3 are of the form $(0, \lambda, \mu, 0)$.

[2 marks] *Standard exercise.*

(c) In particular, the matrix M is diagonalizable, since we can find a basis of four linearly independent eigenvectors, e.g. $((1, 0, 0, -1), (0, 0, 1, -1), (0, 1, 0, 0), (0, 0, 1, 0))$.

[2 marks] *Standard exercise.*

10 marks in total for Question 4

5. We compute:

$$\begin{aligned} f(u_1, u_1) &= 2 \cdot 2 + 2 \cdot 1 \cdot 2 + 1 \cdot 1 = 9, \\ f(u_1, u_2) &= 2 \cdot (-1) + 2 \cdot 1 \cdot (-1) + 1 \cdot 2 = -2, \\ f(u_2, u_1) &= (-1) \cdot 2 + 2 \cdot 2 \cdot 2 + 2 \cdot 1 = 8, \\ f(u_2, u_2) &= (-1) \cdot (-1) + 2 \cdot 2 \cdot (-1) + 2 \cdot 2 = 1, \end{aligned}$$

So, the matrix of f wrt u_1, u_2 is

$$A = \begin{pmatrix} 9 & -2 \\ 8 & 1 \end{pmatrix}.$$

[3 marks]

Similarly,

$$\begin{aligned} f(v_1, v_1) &= 1 \cdot 1 + 2 \cdot 3 \cdot 1 + 3 \cdot 3 = 16, \\ f(v_1, v_2) &= 1 \cdot 0 + 2 \cdot 3 \cdot 0 + 3 \cdot 5 = 15, \\ f(v_2, v_1) &= 0 \cdot 1 + 2 \cdot 5 \cdot 1 + 5 \cdot 3 = 25, \\ f(v_2, v_2) &= 0 \cdot 0 + 2 \cdot 5 \cdot 0 + 5 \cdot 5 = 25, \end{aligned}$$

So the matrix of f wrt v_1, v_2 is $B = \begin{pmatrix} 16 & 15 \\ 25 & 25 \end{pmatrix}$.

[3 marks]

To compute the change-of-basis matrix, we write v_j as linear combinations of the u_j. (Again, this will involve solving a system of linear equations.)

$$\begin{aligned} (1, 3) &= 1 \cdot (2, 1) + 1 \cdot (-1, 2) \\ (0, 5) &= 1 \cdot (2, 1) + 2 \cdot (-1, 2). \end{aligned}$$

So the change-of-basis matrix is $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Alternatively, we can obtain P as the composition of change-of-basis matrices from the given bases to the standard basis:

$$P = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 5 & 5 \\ 5 & 10 \end{pmatrix}.$$

Finally, it is easily checked that

$$P^T A P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 9 & -2 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = B.$$

[3 marks].

9 marks in total for Question 5

Whole question: seen similar in exercises.

6. The *rank* of φ is the dimension of $\mathrm{Im}(\varphi)$. The *nullity* of φ is the dimension of $\ker(\varphi)$.

[1 mark]. *Standard definitions from lectures.*

The rank and nullity theorem states that

$$\dim V = \mathrm{rank}(\varphi) + \mathrm{nullity}(\varphi).$$

[1 mark]. *Standard theorem from lectures.*

For $v_1 = a_1x^2 + b_1x + c_1$ and $v_2 = a_2x^2 + b_2x + c_2$ and $\lambda, \mu \in \mathbb{R}$, we have

$$\begin{aligned}
&\varphi(\lambda v_1 + \mu v_2)\\
&= ((\lambda a_1 + \mu a_2 + \lambda c_1 + \mu c_2, -2(\lambda c_1 + \mu c_2) + \lambda b_1 + \mu b_2 - 2(\lambda a_1 + \mu a_2), 3(\lambda b_1 + \mu b_2))\\
&= \lambda(a_1 + c_1, -2c_1 + b_1 - 2a_1, 3b_1) + \mu(a_2 + c_2, -2c_2 + b_2 - 2a_2, 3b_2) = \lambda\varphi(v_1) + \mu\varphi(v_2).
\end{aligned}$$

Thus φ is linear.

[2 marks]. *Standard exercise.*

There are several ways of determining the rank and nullity; usually we would want to use the rank and nullity theorem. For example, consider an arbitrary polynomial $v = ax^2 + bx + c$ in V. Then $v \in \ker(\varphi)$ if and only if

$$a + c = 0, \quad -2c + b - 2a = 0 \quad \text{and} \quad 3b = 0,$$

which is clearly the case if and only if $b = 0$ and $a = -c$. So

$$\ker(\varphi) = \{ax^2 - a : a \in \mathbb{R}\}.$$

So $\mathrm{nullity}(\varphi) = 1$. Consequently $\mathrm{rank}(\varphi) = \dim(V) - \mathrm{nullity}(\varphi) = 3 - 1 = 2$.

[4 marks]. *Standard exercise.*

Since $\mathrm{nullity}(\varphi) \neq 0$, φ is not an isomorphism.

[1 mark]. *Standard exercise.*

(*Remark:* We have $\mathrm{Im}(\varphi) = \{(a, b, c) : 6a + 3b = c\}$.)

9 marks in total for Question 6

Section B

7. The matrix of the quadratic form

$$q(x, y, z) = 3x^2 - y^2 - 3z^2 + 8xz.$$

with respect to the standard bases is

$$A = \begin{pmatrix} 3 & 0 & 4 \\ 0 & -1 & 0 \\ 4 & 0 & -3 \end{pmatrix}.$$

[3 marks].

We can find a basis with respect to which q is diagonal by finding a basis consisting of orthogonal eigenvectors of A. The characteristic polynomial is

$$\begin{aligned} \det(\lambda I - A) &= \left| \begin{pmatrix} (\lambda - 3) & 0 & -4 \\ 0 & (\lambda + 1) & 0 \\ -4 & 0 & (\lambda + 3) \end{pmatrix} \right| \\ &= (\lambda + 1) \left| \begin{pmatrix} (\lambda - 3) & -4 \\ -4 & (\lambda + 3) \end{pmatrix} \right| \\ &= (\lambda + 1)(\lambda^2 - 9 - 16) \\ &= (\lambda + 1)(\lambda - 5)(\lambda + 5), \end{aligned}$$

so the eigenvalues are -1, 5, -5. Solving the corresponding linear equations gives eigenvectors $(0, 1, 0)$, $(2, 0, 1)$ and $(1, 0, -2)$. The desired matrix P is thus given by

$$P = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}.$$

The desired diagonal matrix is

$$D = P^T AP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -25 \end{pmatrix}.$$

[9 marks].

The diagonal matrix has full rank, so the rank of q is 3. The signature is the number of positive entries minus the number of negative entries, and is thus -1. The surface is a hyperboloid of two sheets.

[3 marks].

15 marks in total for Question 7 *Seen somewhat similar in exercises.*

8.

(a) Statement (ii) is true. Indeed, we have

$$b = eb = (a^{-1}a)b = a^{-1}(ab) = a^{-1}(ac) = (a^{-1}a)c = ec = c.$$

[2 marks]. *Seen in Lectures.*

Statement (i) follows from (ii), letting $c = e$. (Alternatively, it can be proved in the same way as (ii).)

[2 marks]. *Seen in Lectures.*

Statement (iii) is false. For example, let $G = C_2$, and let a be the unique non-identity element. Then $a^2 = e$, but $a \neq e$.

[3 marks]. *Unseen.*

(b) First of all, since $ED = E$, D must be the identity element of the group. So we can fill in the corresponding column and row:

*	A	B	C	D	E
A	?	D	?	A	?
B	?	C	?	B	A
C	?	?	A	C	?
D	A	B	C	D	E
E	?	?	?	E	?

Every line and column in the group table must contain each element. The second column is only missing elements A and E; however, the last row already contains an E. So we can complete this column:

*	A	B	C	D	E
A	?	D	?	A	?
B	?	C	?	B	A
C	?	E	A	C	?
D	A	B	C	D	E
E	?	A	?	E	?

To continue, we can observe, for example, that $BA = BEB = AB = D$. This allows us to fill in the second row.

*	A	B	C	D	E
A	?	D	?	A	?
B	D	C	E	B	A
C	?	E	A	C	?
D	A	B	C	D	E
E	?	A	?	E	?

It is now easy to fill in the rest of the group table:

*	A	B	C	D	E
A	E	D	B	A	C
B	D	C	E	B	A
C	B	E	A	C	D
D	A	B	C	D	E
E	C	A	D	E	B.

[5 marks]. *Seen similar in exercises.*

(c) The cyclic group C_5 with five elements has the same group table.

[3 marks]. *Unseen.*

15 marks in total for Question 8

9. Let $V = \mathbb{R}^{2\times 2}$, and let

$$U := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b + c = 0 \text{ and } a + d = 0 \right\}.$$

$$W := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b + c = 0,\ a + c = 0 \text{ and } a = b \right\}.$$

Let $v_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ be elements of U. Then

$$\lambda v_1 + \mu v_2 = \begin{pmatrix} \lambda a_1 + \mu a_2 & \lambda b_1 + \mu b_2 \\ \lambda c_1 + \mu c_2 & \lambda d_1 + \mu d_2 \end{pmatrix},$$

and we see that

$$(\lambda c_1 + \mu c_2) + (\lambda d_1 + \mu d_2) = \lambda(c_1 + d_1) + \mu(c_2 + d_2) = \lambda \cdot 0 + \mu \cdot 0 = 0, \quad \text{and}$$
$$(\lambda a_1 + \mu a_2) + (\lambda d_1 + \mu d_2) = \lambda(a_1 + d_1) + \mu(a_2 + d_2) = 0;$$

i.e., $v_1 + v_2 \in U$. Thus U is a subspace of V. **[4 marks].**

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U$, then we have $d = -a$ and $c = -b$, so we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so the two vectors

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$$

form a spanning set of U. Since they are also clearly linearly independent, they form a basis of U, and thus the dimension of U is two.

Similarly, the vectors

$$\left(\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

form a basis for W, and the dimension of W is also two.

[4 marks].

Since the vector $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ belongs to both U and W, we see that the dimension of $U \cap W$ is at least 1; since $U \neq W$, the dimension is also at most 1. In particular, the above vector forms a basis for $U \cap W$.

(Alternatively, we can solve the equations in the definitions of U and W simultaneously, and obtain $b = a$, $c = -a$ and $d = -a$.) **[3 marks].**

We thus have $\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 2 + 2 - 1 = 3$. A basis for $U + W$ is given by the three vectors

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

which are clearly linearly independent.

[3 marks].

Since $U \cap W \neq \{0\}$, V is not the direct sum of U and W.

[1 mark].

15 marks in total for Question 9 *Seen similar in exercises*

10.

(a) We have

$$\begin{aligned}\ker(\varphi) &= \{(x,y,z) : y = -4x, z = 2x\} \\ &= \{(x,-4x,2x) : x \in \mathbb{R}\}.\end{aligned}$$

A basis for this space is given by $(1,-4,2)$, so $\text{nullity}(\varphi) = 1$.

[3 marks]. *Standard exercise.*

In particular, we see that $\text{rank}(\varphi) = 3 - 1 = 2$, so we only need to find two linearly independent vectors in the image of φ. Two such vectors are given by $v_1 = \varphi(0,1,0) = (0,1,0)$ and $v_2 = \varphi(0,0,1) = (0,0,1)$.

(Any basis of $\text{Im}(\varphi) = \{(0,y,z) : y,z \in \mathbb{R}\}$ gives a correct answer.)

[3 marks]. *Standard exercise.*

(b) We have already seen that the vectors $\{(0,y,z) : y,z \in \mathbb{R}\}$ are eigenvectors of φ with eigenvalue 1, and that the vectors $(x,-4x,2x)$ are eigenvectors with eigenvalue 0. Since the dimensions of these spaces add up to three, these are all the eigenvectors and eigenvalues of φ.

(Alternatively, write down the matrix of φ with respect to the standard basis, and find all the eigenvectors and eigenvalues by solving the characteristic equation, etc.) **[3 marks]**.

(c) Since we have found three linearly independent eigenvectors, φ is diagonalizable, so the Jordan normal form of φ is diagonal. A basis which diagonalizes φ is given by any basis consisting of three linearly independent eigenvectors; e.g.

$$B = ((1,-4,2),(0,1,0),(0,0,1)).$$

[3 marks].

It is now easy to compute the matrix A, and verify that indeed

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

as required.

[3 marks]. *Similar example seen on exercise sheet.*

15 marks in total for Question 10