THE UNIVERSITY
of LIVERPOOL

SECTION A

1.

(a) Let V be a vector space. Define the subspace span(A) *spanned* by a subset $A = \{v_1, \ldots, v_k\}$ of V.

(b) Let $V = \mathbb{R}^3$, and let U be the subspace of V spanned by $u_1 = (3, 0, 1)$, $u_2 = (1, -2, 1)$ and $u_3 = (1, 4, -1)$. Find the dimension of U.

(c) Let W be the subspace of V spanned by $w_1 = (2, 2, 0)$, $w_2 = (-4, 2, -2)$ and $w_3 = (5, 2, 1)$. Show that $U = W$.

[9 marks]

2. Define the terms: *group, homomorphism, injective, surjective.*

Let G be the group of non-zero real numbers under multiplication. Let H be the group of invertible 2×2 matrices with real entries, under matrix multiplication. [You need not show that these are groups]. Let $\varphi : G \to H$ be defined by

$$\varphi(x) = \begin{pmatrix} x & 0 \\ 0 & x^2 \end{pmatrix}.$$

Is φ a homomorphism? Is φ injective? Is φ surjective? (Justify your answers.)

[9 marks]

3. Let G be a group.

(a) Say what it means for a function $\varphi : G \to G$ to be an *isomorphism* of G to itself.

(b) Show that the set of isomorphisms from G to itself is a group under composition.

[9 marks]

4. Let $V = \mathbb{R}^4$, and let the linear map $\varphi : V \to V$ be defined by

$$\varphi(a, b, c, d) = (-a, 3b, 4(a + d) + 3c, -d).$$

[You need not show that φ is linear.]

(a) Find M, the matrix representation of φ with respect to the standard basis.

(b) What are the eigenvalues and eigenvectors of M?

(c) Is M diagonalizable?

[10 marks]

5. Let f be the bilinear form on $\mathbb{R}^2$ defined by

$$f\big((x_1, y_1), (x_2, y_2)\big) = x_1x_2 + 2y_1x_2 + y_1y_2.$$

Let $u_1 = (2, 1), u_2 = (-1, 2)$. Compute $f(u_1, u_1), f(u_1, u_2), f(u_2, u_1), f(u_2, u_2)$. Find the matrix A of f relative to the basis $\{u_1, u_2\}$. Find the matrix B of f relative to the basis $\{v_1, v_2\}$, where $v_1 = (1, 3), v_2 = (0, 5)$.

Find the change of basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ and check that $B = P^TAP$. [9 marks]

6. Let V and W be finite-dimensional vector spaces, and let $\varphi : V \to W$ be linear. Define the *rank* and *nullity* of φ. State the rank and nullity theorem.

Let $V = \mathrm{Pol}_2(\mathbb{R})$ be the vector space of polynomials of degree at most 2, and let $W = \mathbb{R}^3$. Define $\varphi : V \to W$ by

$$\varphi(ax^2 + bx + c) = (a + c, -2c + b - 2a, 3b).$$

Show that φ is a linear map and compute its rank and nullity. Is φ an isomorphism? [9 marks]

SECTION B

7. Consider the quadratic form

$$q(x, y, z) = 3x^2 - y^2 - 3z^2 + 8xz.$$

Give the matrix A representing q with respect to the standard basis. Find a diagonal matrix D equivalent to A and the matrix P which describes the change of basis from the standard basis to the basis in which q is diagonal. What are the rank and signature of q? What type of quadric surface is described by the equation $q(x, y, z) = 10$? [15 marks]

8.

(a) Let G be a group, and let e be the identity element of G. Which of the following statements are true, and which are false? Give proofs of any true statements, and counterexamples to any false statements.

(i) Let $a, b \in G$, and suppose that $ab = a$. Then $b = e$.

(ii) Let $a, b, c \in G$, and suppose that $ab = ac$. Then $b = c$.

(iii) Let $a \in G$ and suppose that $a^2 = e$. Then $a = e$.

(b) The following is a partially completed group table. Fill in the missing entries. You must justify why each choice of entry is the only one possible.

*	A	B	C	D	E
A	?	D	?	?	?
B	?	C	?	?	A
C	?	?	A	?	?
D	?	?	?	?	?
E	?	?	?	E	?

(c) Name a known group with the group table from (b).

[15 marks]

9. Let $V = \mathbb{R}^{2\times 2}$ be the vector space of real 2×2 matrices. Let

$$U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b + c = 0 \text{ and } a + d = 0 \right\}.$$

Show that U is a subspace of V.

Similarly,

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b + c = 0,\ a + c = 0 \text{ and } a = b \right\}.$$

is a subspace of V [you do not need to show this].

Find bases for each of $U, W, U \cap W$ and $U + W$. Is it true or false that $U + W = U \oplus W$? [15 marks]

10. Consider the linear map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\varphi(x, y, z) = (0, y + 4x, z - 2x).$$

[You need not show that φ is linear.]

(a) Find a basis for U, the image of φ, and a basis for W, the kernel of φ. What are the rank and the nullity of φ?

(b) Find the eigenvalues and eigenvectors of φ.

(c) Find a basis B of $\mathbb{R}^3$ such that the matrix A of φ with respect to B is in Jordan normal form. (When you have found B, you should compute the matrix A to verify that it has the required form.)