SECTION A

1. The set $\{v_1, \dots v_k\}$ spans V if every $v \in V$ can be written as a linear combination $v = \lambda_1 v_1 + \dots \lambda_k v_k$, for some $\lambda_1, \dots, \lambda_k \in \mathbf{R}$.

[2 marks]. *Definition from lectures.*

First put u_1, u_2, u_3 as the rows of a matrix, and use row operations to reduce to echelon form:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ 2 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the space U is spanned by $\{(1,0,-2),(0,1,1)\}$ which are clearly linearly independent and so give a basis for U.

Similarly put w_1, w_2, w_3 as the rows of a matrix, and use row operations to reduce to echelon form:

$$\begin{pmatrix} 1 & -1 & -3 \\ 2 & -1 & -5 \\ 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the space W also has the same basis as U, namely: $\{(1,0,-2),(0,1,1)\}$, and so $U = W$.

[7 marks]. *Seen similar in exercises.*

9 marks in total for Question 1

2. A *group* is a set G together with a binary operation $*$ such that: **(1)** for all $g_1, g_2 \in G$, $g_1 * g_2 \in G$; **(2)** for all $g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$; **(3)** there exists an element $e \in G$ such that, for all $g \in G$, $e * g = g * e = g$; **(4)** for every $g \in G$, there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$. If G, H are groups, then a map $\phi : G \to H$ is a *homomorphism* if, for all $g_1, g_2 \in G$, $\phi(g_1 *_1 g_2) = \phi(g_1) *_2 \phi(g_2)$, where $*_1$ is the group law in G and $*_2$ is the group law in H. The *kernel* of ϕ is the set $\{g \in G : \phi(g) = e\}$. The *image* of ϕ is the set $\{\phi(g) : g \in G\}$.

[4 marks]. *Standard definitions from lectures.*

For any $\left(\begin{smallmatrix} a_1 & b_1 \\ 0 & d_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ 0 & d_2 \end{smallmatrix}\right)$ in G, we have

$$\phi\Big(\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}\Big) = \phi\Big(\begin{pmatrix} a_1a_2 & a_1b_2 + b_1d_2 \\ 0 & d_1d_2 \end{pmatrix}\Big) = (a_1a_2)(d_1d_2).$$

$$\phi\Big(\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}\Big)\phi\Big(\begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}\Big) = (a_1d_1)(a_2d_2) = (a_1a_2)(d_1d_2), \text{ also.}$$

Hence, ϕ is a homomorphism.

We also have: $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \ker\phi \iff \phi\Big(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\Big) = 1 \iff ad = 1 \iff d = 1/a$. So

$$\text{kernel of } \phi = \{\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} : a, b \in \mathbf{R}, a \neq 0\}.$$

Finally, the image of ϕ is all of H, since any nonzero $r \in \mathbf{R}$ is (for example) $\phi\Big(\left(\begin{smallmatrix} r & 0 \\ 0 & 1 \end{smallmatrix}\right)\Big)$.

[5 marks]. *Seen somewhat similar in exercises.*

9 marks in total for Question 2

3. Let $e_1 = 1, e_2 = x, e_3 = x^2$. Then $L(e_1) = L(1) = x^2 = 0 \cdot e_1 + 0 \cdot e_2 + 1 \cdot e_3$, so that the first column of the matrix should have entries $0, 0, 1$. Similarly, $L(e_2) = 0 \cdot e_1 + (-1) \cdot e_2 + 0 \cdot e_3$ and $L(e_3) = 1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3$, so that the matrix is:

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

[3 marks]

If we now compute $\det(\lambda I - M) = (\lambda - 1)(\lambda + 1)^2$, we see that the possible eigenvalues are $\lambda = 1, -1$. When $\lambda = 1$, a vector $v = a + bx + cx^2$ is an eigenvector with eigenvalue 1 iff $L(v) = 1 \cdot v$ iff $c - bx + ax^2 = a + bx + cx^2$ iff $a = c$ and $b = -b$ iff $a = c$ and $b = 0$ iff v is of the form $a + ax^2$ $(a \neq 0)$. When $\lambda = -1$, a vector $v = a + bx + cx^2$ is an eigenvector with eigenvalue -1 iff $L(v) = (-1) \cdot v$ iff $c - bx + ax^2 = -a - bx - cx^2$ iff $a = -c$ iff v is of the form $a + bx - ax^2$ (a, b not both 0).

[6 marks] *Seen similar in exercises.*

9 marks in total for Question 3

4. (i) First note that $\sigma_\ell, \sigma_m, \rho_{A,2\alpha}$ all leave A unchanged, so that $\sigma_m\sigma_\ell(A) = A = \rho_{A,2\alpha}(A)$. Now, let B be any point on ℓ distinct from A, let $B' = \sigma_m(B)$ and let n be the line through A and B'. Let the point Q be the intersection of m and the line BB'. Now, $|AQ| = |AQ|$ and $|BQ| = |B'Q|$ and angle AQB equals angle AQB' equals $\pi/2$. So, triangle AQB is congruent to AQB', giving that $|AB| = |AB'|$ and angle QAB' is the same as angle BAQ, namely: α. It follows that $B' = \rho_{A,2\alpha}(B)$. Further, $\sigma_\ell(B) = B$, since B lies on ℓ. So, we've shown that $\sigma_m\sigma_\ell(B) = B' = \rho_{A,2\alpha}(B)$. Similarly, let k be the line through A at angle $-\alpha$ from ℓ, and let C be any point on k distinct from A. By a similar argument to above, $\sigma_m\sigma_\ell(C) = \rho_{A,2\alpha}(C)$. This shows that $\sigma_m\sigma_\ell$ and $\rho_{A,2\alpha}$ agree on the three non-collinear points A, B, C. Since these are isometries, and since any isometry is determined by its effect on 3 non-collinear points, we conclude that $\sigma_m\sigma_\ell = \rho_{A,2\alpha}$, as required [it helps also to draw a quick diagram of the above].

[**5 marks**]. *Bookwork from lectures.*

(ii) Let ℓ be the line through B at angle $-\beta/2$ from m. By part (i) we have $\sigma_m\sigma_\ell = \rho_{B,2(\beta/2)} = \rho_{B,\beta}$, as required. Similarly, let n be the line through C at angle $\gamma/2$ from m. By part (i) again we have $\sigma_n\sigma_m = \rho_{C,2(\gamma/2)} = \rho_{C,\gamma}$, as required. Hence, $\rho_{C,\gamma}\rho_{B,\beta} = (\sigma_n\sigma_m)(\sigma_m\sigma_\ell) = \sigma_n(\sigma_m\sigma_m)\sigma_\ell = \sigma_n\sigma_\ell$. Again using part (i), this must a rotation about the point of intersection of ℓ and n through twice the angle from ℓ to n (note that the given fact, $\beta \neq -\gamma$, guarantees that ℓ and n are not parallel).

[**5 marks**]. *Broadly similar to bookwork from lectures.*

10 marks in total for Question 4

5. We compute: $f(u_1,u_1) = 1\cdot 1 - 1\cdot 1 + 1\cdot 1 = 1$, $f(u_1,u_2) = 1\cdot 0 - 1\cdot(-1) + 1\cdot(-1) = 0$, $f(u_2,u_1) = 0\cdot 1 - 0\cdot 1 + (-1)\cdot 1 = -1$, $f(u_2,u_2) = 0\cdot 0 - 0\cdot(-1) + (-1)\cdot(-1) = 1$. So, the matrix of f wrt u_1, u_2 is $A = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$.

[3 marks]

Similarly, $f(v_1,v_1) = 2\cdot 2 - 2\cdot 2 + 2\cdot 2 = 4$, $f(v_1,v_2) = 2\cdot 0 - 2\cdot 1 + 2\cdot 1 = 0$, $f(v_2,v_1) = 0\cdot 2 - 0\cdot 2 + 1\cdot 2 = 2$, $f(v_2,v_2) = 0\cdot 0 - 0\cdot 1 + 1\cdot 1 = 1$. So, the matrix of f wrt u_1, u_2 is $B = \left(\begin{smallmatrix} 4 & 0 \\ 2 & 1 \end{smallmatrix}\right)$.

[3 marks]

Now, note that $v_1 = 2\cdot u_1 + 0\cdot u_2$, so that "2" and "0" are the entries of the first column of the change-of-basis matrix. Similarly, $v_2 = 0\cdot u_1 + (-1)u_2$, so that "0" and "$-1$" are the entries of the second column of the change-of-basis matrix. This gives $P = \left(\begin{smallmatrix} 2 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ as the required change-of-basis matrix. Finally, check that: $P^TAP = \left(\begin{smallmatrix} 2 & 0 \\ 0 & -1 \end{smallmatrix}\right)^T \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 0 \\ 0 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 0 \\ 0 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 0 \\ -2 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 4 & 0 \\ 2 & 1 \end{smallmatrix}\right) = B$, as required.

[3 marks]. *Whole question: seen similar (once) in exercises.*

9 marks in total for Question 5

6. A matrix M is *orthogonal* if $MM^T = I$. Let $P = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ and $Q = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$. Then

$$(PQ)^T = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}^T = \begin{pmatrix} a_1a_2 + b_1c_2 & c_1a_2 + d_1c_2 \\ a_1b_2 + b_1d_2 & c_1b_2 + d_1d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_2 & c_2 \\ b_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & c_1 \\ b_1 & d_1 \end{pmatrix} = Q^TP^T.$$

[4 marks]

I is orthogonal, since $II^T = I$. If P, Q are orthogonal then $PP^T = I$ and $QQ^T = I$, so that $(PQ)(PQ)^T = (PQ)Q^TP^T = P(QQ^T)P^T = PIP^T = PP^T = I$, so that PQ is also orthogonal. Also, if P is orthogonal, then $P^T = P^{-1}$, so that $P^{-1}(P^{-1})^T = P^{-1}(P^T)^T = P^{-1}P = I$, so that P^{-1} is also orthogonal. Hence, the set of orthogonal 2×2 matrices contains the identity, is closed, contains inverses, and is associative (since matrix multiplication is always associative), and so is a group. **[5 marks]**. *Seen on exercise sheet.*

9 marks in total for Question 6

SECTION B

7. In U, taking $a = b = d = 0$ gives that $0 \in U$. If $u = a + bx + bx^2 + dx^3 \in U$ and $\lambda \in \mathbf{R}$, then $\lambda u = \lambda(a+bx+bx^2+dx^3) = (\lambda a)+(\lambda b)x+(\lambda b)x^2+(\lambda d)x^3 \in U$. Also, if $u_1 = a_1 + b_1 x + b_1 x^2 + d_1 x^3$ and $u_2 = a_2 + b_2 x + b_2 x^2 + d_2 x^3$ are in U then $u_1 + u_2 = (a_1 + b_1 x + b_1 x^2 + d_1 x^3) + (a_2 + b_2 x + b_2 x^2 + d_2 x^3) = (a_1 + a_2) + (b_1 + b_2)x + (b_1 + b_2)x^2 + (d_1 + d_2)x^3 \in U$. Hence U is a subspace of V. Proof that W is a subspace of V is almost identical.

[3 marks]. *Standard.*

Typical member of U is $a + bx + bx^2 + dx^3 = a \cdot 1 + b \cdot (x + x^2) + d \cdot x^3$, so that $1, x + x^2, x^3$ span U. Also, $\lambda_1 \cdot 1 + \lambda_2 \cdot (x + x^2) + \lambda_3 \cdot x^3 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$, so that $1, x + x^2, x^3$ are linearly independent. Hence this gives a basis for U and so U has dimension 3. Similarly, W has basis $\{1, x - x^2, x^3\}$ and so W also has dimension 3.

[4 marks]. *Standard.*

For $a + bx + cx^2 + dx^3$ to be in $U \cap W$, we must have $c = b$ (to be in U) *and* $c = -b$ (to be in W); but $b = -b \iff b = 0$. So, $U \cap W = \{a + dx^3 : a, d \in R\}$. Clearly (shown as above) $1, x^3$ is a basis for $U \cap W$ and so $U \cap W$ has dimension 2.

[3 marks]. *Harder, but seen similar.*

Note that, any $a + bx + cx^2 + dx^3$ in V can be written as, for example, $(\frac{a}{2} + \frac{b+c}{2}x + \frac{b+c}{2}x^2 + \frac{d}{2}x^3) + (\frac{a}{2} + \frac{b-c}{2}x - \frac{b-c}{2}x^2 + \frac{d}{2}x^3)$, where the first term of this sum is in U and the second term is in W. This means that *any* member of V can be written as (member-of-U) + (member-of-W), that is: $U + W = V$, which has dimension 4. [Alternatively: note that $(1,0,0,0) = (1,0,0,0) + (0,0,0,0)$, $(0,1,1,0) = (0,1,1,0) + (0,0,0,0)$, $(0,1,-1,0) = (0,0,0,0) + (0,1,-1,0)$ and $(0,0,0,1) = (0,0,0,0) + (0,0,0,1)$ are four linearly independent members of $U + W$, so that $U + W$ has dimension at least 4, and is a subspace of the 4-dimensional space V, giving that $U + W = V$].

[3 marks]. *Harder. Unseen.*

Finally note that, since $\dim(U \cap W) = 2$, we do *not* have $U \cap W = \{0\}$, and so $V = U \oplus W$ (note that the definition of $V = U \oplus W$ is that *both* $V = U + W$ *and* $U \cap W = \{0\}$).

[2 marks]. *Seen similar in exercises (once).*

15 marks in total for Question 7

8. (i) The *rank* of ϕ is the dimension of the image of ϕ. The *nullity* of ϕ is the dimension of the kernel of ϕ. That rank & nullity theorem states that $\text{rank}(\phi) + \text{nullity}(\phi) = \dim(V)$.

[3 marks] *From lectures.*

(ii) Let A be the matrix of F wrt the basis E_1, E_2, E_3, E_4. We have $F(E_1) = \left(\begin{smallmatrix} 1 & 0 \\ 3 & 0 \end{smallmatrix}\right) = 1 \cdot E_1 + 0 \cdot E_2 + 3 \cdot E_3 + 0 \cdot E_4$, so that the entries of the first column of A are $\begin{smallmatrix} 1 \\ 0 \\ 3 \\ 0 \end{smallmatrix}$. Similarly, we have $F(E_2) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 3 \end{smallmatrix}\right) = 0 \cdot E_1 + 1 \cdot E_2 + 0 \cdot E_3 + 3 \cdot E_4$, which gives the entries of the second column of A. Similarly $F(E_3) = \left(\begin{smallmatrix} 2 & 0 \\ 6 & 0 \end{smallmatrix}\right) = 2 \cdot E_1 + 0 \cdot E_2 + 6 \cdot E_3 + 0 \cdot E_4$, which gives the entries of the third column of A. Finally, $F(E_4) = \left(\begin{smallmatrix} 0 & 2 \\ 0 & 6 \end{smallmatrix}\right) = 0 \cdot E_1 + 2 \cdot E_2 + 0 \cdot E_3 + 6 \cdot E_4$, which gives the entries of the fourth column of A. So, A is: $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 6 & 0 \\ 0 & 3 & 0 & 6 \end{pmatrix}$.

[3 marks]. *Seen similar in exercises.*

Applying column operations to A as follows: $C_3 \to C_3 - 2C_1$ and $C_4 \to C_4 - 2C_2$ gives a matrix which is the same as A, except with all entries zero in the third and fourth columns (and is in column echelon form). The first two columns of A give a basis for the image of F, that is, a basis for the image of F is: $1 \cdot E_1 + 0 \cdot E_2 + 3 \cdot E_3 + 0 \cdot E_4$ and $0 \cdot E_1 + 1 \cdot E_2 + 0 \cdot E_3 + 3 \cdot E_4$, that is to say, a basis for the image of F is: $\{\left(\begin{smallmatrix} 1 & 0 \\ 3 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 3 \end{smallmatrix}\right)\}$. [*Alternative Method:* we could have found a basis for the image of F directly from the definition of F (without needing A) by observing that $F(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a+2c & b+2d \\ 3a+6c & 3b+6d \end{smallmatrix}\right) = (a+2c)\left(\begin{smallmatrix} 1 & 0 \\ 3 & 0 \end{smallmatrix}\right) + (b+2d)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 3 \end{smallmatrix}\right)\}$, so that $\left(\begin{smallmatrix} 1 & 0 \\ 3 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 3 \end{smallmatrix}\right)$ span the image of F, and are clearly linearly independent, and so give a basis for the image of F].

[3 marks]. *Unseen.*

Solving for $A\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, we first apply row operations to A as follows: $R_3 \to 3R_1$ and $R_4 \to R_4 - 3R_2$ gives a matrix which is the same as A except with all entries zero in the last two rows (and is in row echelon form). This gives only two independent equations: $a + 2c = 0$ and $b + 2d = 0$, so take c, d as the two free parameters so that the general solution for a, b, c, d is: $-2c, -2d, c, d$, that is: $-2cE_1 - 2dE_2 + cE_3 + dE_4$. The typical member of the kernel of F is then: $\left(\begin{smallmatrix} -2c & -2d \\ c & d \end{smallmatrix}\right) = c\left(\begin{smallmatrix} -2 & 0 \\ 1 & 0 \end{smallmatrix}\right) + d\left(\begin{smallmatrix} 0 & -2 \\ 0 & 1 \end{smallmatrix}\right)$. So, $\left(\begin{smallmatrix} -2 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -2 \\ 0 & 1 \end{smallmatrix}\right)$ span the kernel of F and are clearly linearly independent. So, $\{\left(\begin{smallmatrix} -2 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -2 \\ 0 & 1 \end{smallmatrix}\right)\}$ is a basis for the kernel of F. [*Alternative Method:* we could have found a basis for the kernel of F directly from the definition of F (without needing A) by observing that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \ker F \iff F(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \iff F(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a+2c & b+2d \\ 3a+6c & 3b+6d \end{smallmatrix}\right) \iff \text{a} + \text{2c} = 0$, $\text{b+2d} = 0 \iff \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = c\left(\begin{smallmatrix} -2 & 0 \\ 1 & 0 \end{smallmatrix}\right) + d\left(\begin{smallmatrix} 0 & -2 \\ 0 & 1 \end{smallmatrix}\right)$, again giving $\{\left(\begin{smallmatrix} -2 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -2 \\ 0 & 1 \end{smallmatrix}\right)\}$ as a basis for the kernel of F.]

Since a basis for the image of F has two elements, it follows that $\text{rank}(F) = 2$. Since a basis for the kernel of F has two elements, it follows that $\text{nullity}(F) = 2$. Also, $\dim(V) = 4$, since $\{E_1, E_2, E_3, E_4\}$ is a basis for V. So, the rank & nullity theorem is verified in this case as: $2 + 2 = 4$.

[6 marks]. *Seen (somewhat) similar in exercises.*

15 marks in total for Question 8

9. Taking the standard matrix A for q, we form $(A|I)$. We then apply to A: $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 3R_1$, $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 + 3C_1$ as step one and $R_3 \to R_3 - 2R_2$, $C_3 \to C_3 - 2C_2$ as step two (with only the column operations being applied to I) to give:

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 1 & 0 \\ -3 & -4 & 8 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 7 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & -5 & 0 & 0 & 1 \end{array}\right) = (D|P).$$

[7 marks].

Then $D = P^T AP$, and $A = Q^T DQ$, where $Q = P^{-1} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. New variables: $\begin{pmatrix} r \\ s \\ t \end{pmatrix} = Q \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (that is, we are changing to new variables r, s, t, where $r = x + 2y - 3z$, $s = y + 2z$, $t = z$) transform $q(x, y, z)$ into $\tilde{q}(r, s, t) = r^2 + s^2 - 5t^2$.

[3 marks]

The rank of q is 3 (which is the number of nonzero entries of D), and the signature of q is the number of positive entries of D minus the number of negative entries $= 2 - 1 = 1$. The surface $q(x, y, z) = 25$ becomes $r^2 + s^2 - 5t^2 = 25$, in r, s, t coordinates, which is a hyperboloid of one sheet. The sketch should look identical to the standard sketch of a hyperboloid of one sheet, except that the x, y, z axes should be labelled r, s, t (if drawn it wrt r, s, t). [If drawn wrt x, y, z then it should be made clear in the diagram that the axes of the surface are: $y = z = 0$, $x + 2y = z = 0$, $x + 2y - 3z = y + 2z = 0$].

[5 marks]. *Whole question: seen similar in exercises.*

15 marks in total for Question 9

10.(i) Suppose that e_1 and e_2 were both (2-sided) identity elements. Then $e_1 * e_2 = e_1$, since e_2 is an identity. Similarly, $e_1 * e_2 = e_2$. Hence $e_1 = e_2$.

[2 marks]. *Seen in lectures.*

Let $\alpha * \beta = e$. Let δ be the (2-sided) inverse of α, and multiply both sides of the equation on the left by δ. Then $\delta * (\alpha * \beta) = \delta * e = \delta$ (since e is identity), so that $(\delta * \alpha) * \beta = \delta$ (assoc.) and so $\beta = \delta$. Now multiply both sides on the right by α, giving $\beta * \alpha = \delta * \alpha = e$.

[2 marks]. *Unseen*

(ii) Suppose $\alpha * \beta = \alpha * \gamma$. Multiply both sides on the left by δ, the inverse of α. Then $\delta * (\alpha * \beta) = \delta * (\alpha * \gamma)$, giving $(\delta * \alpha) * \beta = (\delta * \alpha) * \gamma$ [by associativity], and so $e * \beta = e * \gamma$, finally giving: $\beta = \gamma$, as required. The values of $\alpha * g$, as g runs through all the members of the group give the 'α' row of the group table; if two of these were the same, we would have $\alpha * \beta = \alpha * \gamma$ for distinct $\beta \neq \gamma$, contradicting the previous result. Similarly, $\beta * \alpha = \gamma * \alpha \Rightarrow \beta = \gamma$ gives that no element can be repeated in the same column.

[4 marks]. *Seen on exercise sheet.*

(iii) From the already provided entry $B * F = B$, we deduce (after multiplying both sides on left by the inverse of B) that F is the identity element. This allows us to fill in the bottom row as ABCDEF and similarly the right hand column. Having done this, the 'no-element-repeated-in-the-same-row-or-column' rule excludes A,B,C,D,E from $D * E$ and so the only possibility for $D * E$ is F. But F is the identity element, so by the second part of (i), we have $E * D = F$, also (i.e. D must be the 2-sided inverse of E). At this point we have:

*	A	B	C	D	E	F
A	F	?	?	?	B	A
B	?	?	?	?	C	B
C	?	D	?	?	A	C
D	?	?	?	E	F	D
E	?	?	B	F	?	E
F	A	B	C	D	E	F

From now on, we can fill in all the remaining entries by using only the 'no-element-repeated-in-the-same-row-or-column' rule. For example, this forces $E * E$ to be D. The following gives a possible order in which the remaining 16 entries can be fixed using this rule.

*	A	B	C	D	E	F
A	F	12	16	4	B	A
B	11	13	15	3	C	B
C	10	D	14	2	A	C
D	9	8	5	E	F	D
E	6	7	B	F	1	E
F	A	B	C	D	E	F

The final table must then be

*	A	B	C	D	E	F
A	F	E	D	C	B	A
B	D	F	E	A	C	B
C	E	D	F	B	A	C
D	B	C	A	E	F	D
E	C	A	B	F	D	E
F	A	B	C	D	E	F

[**7 marks**]**.** *Seen similar on Ex Sheet (but this one is harder).*

15 marks in total for Question 10