SECTION A

1. Say what it means for $\{v_1, \ldots, v_k\}$ to *span* a vector space V.

Let U be the subspace of $\mathbf{R}^3$ spanned by

$$u_1 = (1, 1, -1), u_2 = (1, 2, 0), u_3 = (2, 0, -4).$$

Let W be the subspace of $\mathbf{R}^3$ spanned by

$$w_1 = (1, -1, -3), w_2 = (2, -1, -5), w_3 = (1, 2, 0).$$

Show that $U = W$. [9 marks]

2. Define the terms: *group, homomorphism, kernel, image.*

Let $G = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a, b, d \in \mathbf{R}, ad \neq 0\}$, under the operation of matrix multiplication. Let H be the group of nonzero real numbers, under the operation of multiplication [you need not show that these are groups]. Let $\phi : G \to H$ be defined by

$$\phi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = ad.$$

Show that ϕ is a homomorphism. Find the kernel and image of ϕ. [9 marks]

3. Let V be the vector space of polynomials in x of degree at most 2 with coefficients in $\mathbf{R}$. Let the linear map $L : V \to V$ be defined by

$$L(a + bx + cx^2) = c - bx + ax^2.$$

Find M, the matrix representation of L with respect to the basis $\{1, x, x^2\}$. What are the eigenvalues and eigenvectors of M? [9 marks]

4. For any point A and angle α, let $\rho_{A,\alpha}$ denote rotation anticlockwise about A through angle α. For any line ℓ let σ_ℓ denote reflection in the line ℓ.

(i) Let ℓ and m be two lines which both pass through point A. Let α be the angle from ℓ to m. Show that $\sigma_m \sigma_\ell = \rho_{A,2\alpha}$.

(ii) Let B and C be two distinct points, let m be the line through B and C, and let β, γ be any two angles, where $\beta \neq -\gamma$. Use part (i) to find a line ℓ through B such that $\sigma_m \sigma_\ell = \rho_{B,\beta}$. Similarly, find a line n through C such that $\sigma_n \sigma_m = \rho_{C,\gamma}$. Hence, or otherwise, show that $\rho_{C,\gamma} \rho_{B,\beta}$ is a rotation. [10 marks]

5. Let f be the bilinear form on $\mathbf{R}^2$ defined by

$$f((x_1, x_2), (y_1, y_2)) = x_1y_1 - x_1y_2 + x_2y_2.$$

Let $u_1 = (1,1), u_2 = (0,-1)$. Compute $f(u_1, u_1), f(u_1, u_2), f(u_2, u_1), f(u_2, u_2)$. Find the matrix A of f relative to the basis $\{u_1, u_2\}$. Find the matrix B of f relative to the basis $\{v_1, v_2\}$, where $v_1 = (2,2), v_2 = (0,1)$.

Find the change of basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ and show that $B = P^TAP$. [9 marks]

6. Define what it means for a matrix to be *orthogonal*. Let P, Q be 2×2 matrices with real entries; show that $(PQ)^T = Q^TP^T$.

Show that the set of 2×2 orthogonal matrices with real entries is a group under matrix multiplication. [9 marks]

SECTION B

7. Let V be the vector space of polynomials in x of degree at most 3, with real coefficients. Let

$$U = \{a + bx + bx^2 + dx^3 : a, b, d \in \mathbf{R}\}, \quad W = \{a + bx - bx^2 + dx^3 : a, b, d \in \mathbf{R}\}.$$

Show that U and W are subspaces of V. What are the dimensions of each of $U, W, U \cap W$ and $U + W$? Is it true or false that $V = U \oplus W$? [15 marks]

8. (i) Let $f : V \to W$ be a linear map between two vector spaces V and W. Define the *rank* of f and the *nullity* of f. State the rank & nullity theorem.

(ii) Let $V = M_2(\mathbf{R})$, the vector space of 2×2 matrices with real entries, and let $M = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. Let $F : V \to V$ be the linear map defined by $F(A) = MA$. Find the matrix of F with respect to the basis $\{E_1, E_2, E_3, E_4\}$, where E_1, E_2, E_3, E_4 are $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, respectively.

Find a basis for the image of F and a basis for the kernel of F. Find the rank of F and the nullity of F. Verify that the rank & nullity theorem holds in this case. [15 marks]

9. Consider the quadratic form

$$q(x, y, z) = x^2 + 4xy + 5y^2 - 6xz - 8yz + 8z^2.$$

Give the matrix A representing q with respect to the standard basis. Find a diagonal matrix D equivalent to A and the matrix P which describes the change of basis from the standard basis to the basis in which q is diagonal. What are the rank and signature of q? Describe geometrically the surface $q(x, y, z) = 25$. Draw a sketch of the surface. [15 marks]

10.(i) Let G be a group. Show that the identity element e is unique. Show that $\alpha * \beta = e \Rightarrow \beta * \alpha = e$, for any $\alpha, \beta \in G$,

(ii) Show that, for any $\alpha, \beta, \gamma \in G$, $\alpha * \beta = \alpha * \gamma \Rightarrow \beta = \gamma$. Deduce that no element can be repeated in the same row inside a group table. Similarly show that no element can be repeated in the same column.

(iii) The following is a partially completed group table for a group with six elements. Fill in the missing entries. You must justify (entry by entry) why each choice of entry is the only one possible.

*	A	B	C	D	E	F
A	F	?	?	?	B	?
B	?	?	?	?	C	?
C	?	D	?	?	A	?
D	?	?	?	E	?	?
E	?	?	B	?	?	?
F	?	B	?	?	?	?

[15 marks]