Solutions to 1997 May Examination. 2MP44.

SECTION A

1. A *group* is a set G together with a binary operation $*$ such that: **(1)** for all $g_1, g_2 \in G$, $g_1 * g_2 \in G$; **(2)** for all $g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$; **(3)** there exists an element $e \in G$ such that, for all $g \in G$, $e * g = g * e = g$; **(4)** for every $g \in G$, there exists $g^{-1} \in G$ such that $gg^{-1} = g^{-1} * g = e$ **(2 marks)**. If G, H are groups, then a map $\phi : G \to H$ is a *homomorphism* if, for all $g_1, g_2 \in G$, $\phi(g_1 *_1 g_2) = \phi(g_1) *_2 \phi(g_2)$, where $*_1$ is the group law in G and $*_2$ is the group law in H **(1 mark)**. The map ϕ is *injective* if, for all $g_1, g_2 \in G$, $\phi(g_1) = \phi(g_2) \Rightarrow g_1 = g_2$ **(1 mark)**. The map ϕ is *surjective* if, for all $h \in H$, there exists $g \in G$ such that $\phi(g) = h$ **(1 mark)**. The given map is indeed homomorphism since, for any $x_1, x_2 \in \mathbf{Z}$, $f(x_1 + x_2) = 2^{x_1 + x_2} = 2^{x_1} 2^{x_2} = f(x_1) f(x_2)$ **(1 mark)**. The given map is injective since, for any $x_1, x_2 \in \mathbf{Z}$, if $f(x_1) = f(x_2)$ then $2^{x_1} = 2^{x_2}$ and so, taking log base 2 of both sides, $x_1 = x_2$ **(2 marks)**. The given map is not surjective since, for example, $3 \notin \operatorname{im} f$ – using the fact that $f(x) = 2^x$, $x \in \mathbf{Z}$, must be either ≤ 1 (when $x \leq 0$) or an integer divisible by 2 (when $x \geq 1$) **(1 mark)**. [**Total for question 1: 9 marks**]

2. A finite set of vectors $S = \{v_1, \ldots v_n\}$ is a *basis* for V if: **(1)** S spans V – that is, every $v \in V$ can be written as a finite linear combination of members of S; **(2)** S is linearly independent – that is, whenever $\lambda_1 v_1 + \ldots \lambda_n v_n = 0$ then $\lambda_1 = \ldots = \lambda_n = 0$ **(4 marks)**. For the given set, if we write the vectors wrt the standard basis $1, x, x^2, x^3$, they are: $(0,1,1,1), (1,0,1,1), (1,1,0,1), (1,1,1,0)$. Putting these as the rows of a 4×4 matrix, we can use a few elementary row operations to obtain the identity matrix, so that the given set is a basis **(5 marks)**. [**Total for question 2: 9 marks**]

3. The map $\phi : V \to W$ is a *linear map* if, **(1)** for all $v_1, v_2 \in V$, $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$; **(2)** for all $v \in V, \lambda \in \mathbf{R}$, $\phi(\lambda v) = \lambda \phi(v)$ **(2 marks)**. The *rank* of ϕ is the dimension of the image of ϕ **(1 mark)**. The *nullity* of ϕ is the dimension of the kernel of ϕ **(1 mark)**.

Applying column operations to the standard matrix for ϕ:

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix}. \ (\mathbf{1\ mark})$$

The image of ϕ is the span of the column space, which has basis given by the nonzero columns of the right hand matrix: $\{(1,0,1,1), (0,1,1,2)\}$ **(1 mark)**. The rank of ϕ is therefore the number of elements in this basis, which is 2 **(1 mark)**.

Applying row operations to the standard matrix for ϕ:

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A, \text{ say. } (\mathbf{1\ mark})$$

So, $(x, y, z) \in \ker \phi$ iff $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ iff $x + z = 0, y + z = 0$, which has general solution: $(x, y, z) = (-z, -z, z) = z(-1, -1, 1)$, and so $\{(-1, -1, 1)\}$ gives a basis for $\ker \phi$ **(1 mark)**. The nullity of ϕ is therefore the number of elements in this basis, which is 1 **(1 mark)**. [**Total for question 3: 10 marks**]

4. First, we note the following result (from lectures).

Result (*). For any two lines ℓ and m which both pass through point A, we have $\sigma_m \sigma_\ell = \rho_{A,2\theta}$, where θ is the angle from ℓ to m.

Proof (from lectures). First note that $\sigma_\ell, \sigma_m, \rho_{A,2\alpha}$ all leave A unchanged, so that $\sigma_m \sigma_\ell(A) = A = \rho_{A,2\alpha}(A)$. Now, let B be any point on ℓ distinct from A and let $B' = \sigma_m(B)$. Let the point Q be the intersection of m and the line BB'. Now, $|AQ| = |AQ|$ and $|BQ| = |B'Q|$ and angle AQB equals angle AQB' equals $\pi/2$. So, tringle AQB is congruent to AQB', giving that $|AB| = |AB'|$ and angle QAB' is the same as angle BAQ, namely: α. It follows that $B' = \rho_{A,2\alpha}(B)$. Further, $\sigma_\ell(B) = B$, since B lies on ℓ. So, we've shown that $\sigma_m \sigma_\ell(B) = B' = \rho_{A,2\alpha}(B)$. Similarly, let k be the line through A at angle $-\alpha$ from ℓ, and let C be any point on k distinct from A. By a simlar argument to above, $\sigma_m \sigma_\ell(C) = \rho_{A,2\alpha}(C)$. This shows that $\sigma_m \sigma_\ell$ and $\rho_{A,2\alpha}$ agree on the three non-collinear points A, B, C. Since these are isometries, and since any isometry is determined by its effect on 3 non-collinear points, we conclude that $\sigma_m \sigma_\ell = \rho_{A,2\alpha}$, as required (it helps also to draw a quick diagram of the above). [Since the question did not explicitly ask for a proof of Result (*), it can be quoted from lectures. It is prudent to include the proof of Result (*) in your answer, though, since I would give some partial credit for it, even if the rest of your answer made no further progess].

Now, returning to the given exam question, we first note that, if we let r be the line through A at angle $-\alpha/2$ from from n, then by Result (*) we have $\sigma_n \sigma_r = \rho_{A,2(\alpha/2} = \rho_{A,\alpha}$. Similarly, if we let t be the line through A at angle $\alpha/2$ from from n, then by Result (*) we have $\sigma_t \sigma_n = \rho_{A,2(\alpha/2} = \rho_{A,\alpha}$. So, $\rho_{A,\alpha}\sigma_n = \sigma_n \rho_{A,\alpha} \iff (\sigma_t \sigma_n)\sigma_n = \sigma_n(\sigma_n \sigma_r) \iff \sigma_t(\sigma_n \sigma_n) = (\sigma_n \sigma_n)\sigma_r \iff \sigma_t = \sigma_r \iff t = r \iff$ the angle between r and t is 0 or $\pi \iff \alpha/2 + \alpha/2 = 0$ or π [since the angle from r to t is the "angle from r to s plus angle from s to t] $\iff \alpha = 0$ or π, as required. [Note that this is all the same as solution to Ex Sheet 3, Qn 3, with lines ℓ, m, n there corresponding to r, n, t here]. [**Total for question 4: 10 marks**]

5. The vectors u_1, u_2, u_3, expressed in terms of the basis $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, are given by $u_1 = (1, -1, -3, 0), u_2 = (2, -1, -5, 0), u_3 = (1, 2, 0, 0)$. Putting these as rows of a matrix, and using row operations to reduce to echelon form:

$$\begin{pmatrix} 1 & -1 & -3 & 0 \\ 2 & -1 & -5 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero rows $(1, -1, -3, 0), (0, 1, 1, 0)$ of the second-last step above, which correspond to the two matrices given in the question, are clearly linearly independent and are a basis for U, as required **(4 marks)**.

Representing v_1, v_2, v_3 in the same way, putting them as rows of a matrix, and reducing to echelon form gives:

$$\begin{pmatrix}1&2&0&0\\2&3&-1&0\\3&2&-4&0\end{pmatrix}\sim\begin{pmatrix}1&2&0&0\\0&-1&-1&0\\0&-4&-4&0\end{pmatrix}\sim\begin{pmatrix}1&2&0&0\\0&-1&-1&0\\0&0&0&0\end{pmatrix}\sim\begin{pmatrix}1&2&0&0\\0&1&1&0\\0&0&0&0\end{pmatrix}\sim\begin{pmatrix}1&0&-2&0\\0&1&1&0\\0&0&0&0\end{pmatrix}$$

We can see that $(1,0,-2,0)$, $(0,1,1,0)$ is a basis for V **(4 marks)**. Since it is also a basis for U we have that $U = V$ **(1 mark)**. [**Total for question 5: 9 marks**]

6. A *bilinear form* is a map $f : V \times V \to \mathbf{R}$ which satisfies: **(1)** for all $u_1, u_2, v \in V$, $a, b \in \mathbf{R}$, $f(au_1+bu_2, v) = af(u_1, v) + bf(u_2, v)$; **(2)** for all $u, v_1, v_2 \in V$, $a, b \in \mathbf{R}$, $f(u, av_1 + bv_2) = af(u, v_1) + bf(u, v_2)$ **(3 marks)**. Such a map is *symmetric* if, for all $u, v \in V$, $f(u, v) = f(v, u)$ **(2 marks)**. The given map is not a bilinear form since, for example, taking $u_1 = (1,0), u_2 = (1,0), a = 1, b = 1, v = (0,0)$, we have $\phi(au_1 + bu_2, v) = \phi\big((2,0),(0,0)\big) = 4$, whereas $af(u_1,v) + bf(u_2,v) = 1 \cdot f\big((1,0),(0,0)\big) + 1 \cdot f\big((1,0),(0,0)\big) = 1 + 1 = 2$, so that property **(1)** above is not always satisfied **(3 marks)**. [**Total for question 6: 8 marks**]

SECTION B

7. We take A, the matrix representing the quadratic form $f(x, y, z)$, form $(A|I)$, and then use row & column opertations $R_2 \to R_2 + R_1$ & $C_2 \to C_2 + C_1$ followed by: $R_3 \to R_3 - (1/2)R_2$ $C_3 \to C_3 - (1/2)C_2$, with only the column operations being performed on I, as follows:

$$\left(\begin{array}{ccc|ccc}1&-1&0&1&0&0\\-1&3&1&0&1&0\\0&1&3&0&0&1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1&0&0&1&1&0\\0&2&1&0&1&0\\0&1&3&0&0&1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1&0&0&1&1&-\frac{1}{2}\\0&2&0&0&1&-\frac{1}{2}\\0&0&\frac{5}{2}&0&0&1\end{array}\right).$$

If we now let

$$A = \begin{pmatrix}1&-1&0\\-1&3&1\\0&1&3\end{pmatrix},\ D = \begin{pmatrix}1&0&0\\0&2&0\\0&0&\frac{5}{2}\end{pmatrix},\ P = \begin{pmatrix}1&1&-\frac{1}{2}\\0&1&-\frac{1}{2}\\0&0&1\end{pmatrix},\ Q = P^{-1} = \begin{pmatrix}1&-1&0\\0&1&\frac{1}{2}\\0&0&1\end{pmatrix},$$

then $D = P^TAP$ and $A = Q^TDQ$ **(6 marks)**. Here, A represents the quadratic form wrt x, y, z and D represents it wrt new variables r, s, t given by $\left(\begin{smallmatrix}r\\s\\t\end{smallmatrix}\right)=Q\left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)$, that is: $r = x-y$, $s = y+z/2$, $t = z$ **(3 marks)**. Then, $f(x,y,z) = 5$ becomes $g(r,s,t) = r^2 + 2s^2 + (5/2)t^2$ and so the equation for the surface becomes $r^2 + 2s^2 + (5/2)t^2 = 5$ **(2 marks)**. All coefficients are positive, and so this is an ellipsoid. The sketch may be rough, as long as it looks roughly egg-shaped; there must be some indication of orientation; i.e. that the principal axes are the r-axis, s-axis and t-axis, if drawn wrt r, s, t, or the x-axis, the line $x - y = z = 0$ and the line $x - y = y + z/2 = 0$, if drawn wrt x, y, z coordinates **(4 marks)**. [**Total for question 7: 15 marks**]

8. The dual space V^* is defined to be the set of all linear maps from V to $\mathbf{R}$ **(1 mark)**. Given $\theta, \phi \in V^*$, we can define $\theta + \phi$ by: $(\theta + \phi)(x) = \theta(x) + \phi(x)$, for all $x \in V$. Similarly, for $\lambda \in \mathbf{R}$, define $\lambda\theta$ by $(\lambda\theta)(x) = \lambda\big(\theta(x)\big)$, for all $x \in V$ **(2 marks)**. Given a basis $\{x_1, \ldots, x_n\}$ for V, the i-th member of the dual basis, ϕ_i, is defined to be the unique linear map from V to $\mathbf{R}$ such that $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$, for all $j \neq i$ **(2 marks)**. Suppose $f \in V^*$; define $\lambda_j = f(x_j)$ for all j; then $(\lambda_1\phi_1 + \ldots + \lambda_n\phi_n)(x_j) = \lambda_j \cdot \phi_j(x_j)$ [since $\phi_i(x_j) = 0$, for all $j \neq i$] $= \lambda_j$ [since $\phi_j(x_j) = 1$]. Hence, f and $\lambda_1\phi_1 + \ldots + \lambda_n\phi_n$ both take the same values on each of $x_1, \dots x_n$, giving that $f = \lambda_1\phi_1 + \dots + \lambda_n\phi_n$ [since any linear map is completely determined by its values on a basis]. Hence, $\{\phi_1, \dots \phi_n\}$ spans V^*. Now suppose that $\lambda_1\phi_1 + \dots + \lambda_n\phi_n = 0$ for some $\lambda_1, \dots, \lambda_n$. Then, for any j, $(\lambda_1\phi_1 + \dots + \lambda_n\phi_n)(x_j) = 0$, and so $\lambda_j \cdot 1 = 0$; hence $\lambda_1 = \dots = \lambda_n = 0$, and so $\phi_1, \dots \phi_n$ are linearly independent. Hence $\{\phi_1, \dots \phi_n\}$ is a basis for V^*. **(4 marks)**.

In the given example, we want $\phi_1((x,y)) = ax + by \in V^*$ to satisfy $\phi_1(v_1) = 1$ and $\phi_1(v_2) = 0$; that is: $a + b = 1$ and $a + 2b = 0$, which has solution: $a = 2, b = -1$, so that ϕ_1 is defined by: $\phi_1\big((x,y)\big) = 2x - y$. Similarly, we want $\phi_2((x,y)) = cx + dy \in V^*$ to satisfy $\phi_2(v_1) = 0$ and $\phi_2(v_2) = 1$; that is: $c + d = 0$ and $c + 2d = 1$, which has solution: $a = -1, b = 1$, so that ϕ_2 is defined by: $\phi_2\big((x,y)\big) = -x + y$. Hence, $\phi_1\big((2,1)\big) = 3$ and $\phi_2\big((2,1)\big) = -1$. **(6 marks)**. **[Total for question 8: 15 marks]**

9. The vector $v \in V$ is an *eigenvector* of f with *eigenvalue* $\lambda \in \mathbf{R}$ if $v \neq 0$ and $f(v) = \lambda v$ **(3 marks)**. First note that the given map f is just $f : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right)$. One method is to notice that $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and any nonzero linear combination of these, are eigenvectors with eigenvalue 1. Similarly, $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix}\right)$, and any nonzero linear combination of these, are eigenvectors with eigenvalue -1. Since we have now found two eigenspaces each of dimension 2, and since the dimension of the whole vector space in 4, we must have found them all **(12 marks)**. Alternatively, a more time consuming approach is to compute the 4×4 matrix B representing the map f with respect to the standard basis, and then use $\det(\lambda I - B)$, etc.
[Total for question 9: 15 marks]

10. H is a *subgroup* of G if H is a subset of G, $e \in H$ and H forms a group under the same operation as G [alternatively: H is a *subgroup* of G if H is a nonempty subset of G satisfying $h_1h_2^{-1} \in H$ for every $h_1, h_2 \in H$] **(3 marks)**. Lagrange's theorem says that, if G is a finite group, then the order of H divides the order of G **(3 marks)**. A presentation of the given group G is $\langle \sigma, \rho | \sigma^2 = \rho^n = e, \rho\sigma = \sigma\rho^{-1} \rangle$. Here, σ is a fixed reflection, and ρ is a rotation $2\pi/n$ **(3 marks)**. The set H is a subgroup, since e is a rotation of zero degrees, the product of rotations of angles α, β is again a rotation (of angle $\alpha + \beta$), and the inverse of a rotation of angle α is again a rotation (of angle $-\alpha$) **(2 marks)**. The size of H is n and the size of G is $2n$; if there were a K as described, then by Lagrange's Theorem n would have to divide but not equal order(K), and order(K) would have to divide but not equal $2n$, which is impossible; so no such K exists **(4 marks)**.
[Total for question 10: 15 marks]