SECTION A

1. Define the terms: *group, homomorphism, injective, surjective.*

Let G be the group of integers under addition, and let H be the group of nonzero rational numbers under multiplication. Show that the map f, defined by $f(x) = 2^x$, is a homomorphism from G to H. State, giving reasons, whether f is injective. State, giving reasons, whether f is surjective. [9 marks]

2. Define what it means for a finite set of vectors to be a *basis* for a vector space V. Let V be the vector space of polynomials of degree at most 3, with coefficients in $\mathbf{R}$. Show that the set $\{x+x^2+x^3, 1+x^2+x^3, 1+x+x^3, 1+x+x^2\}$ is a basis for V. [9 marks]

3. Define what it means for $\phi : V \to W$ to be a *linear map* between two vector spaces V and W. Define the *rank* and *nullity* of ϕ.

Let $\phi : \mathbf{R}^3 \to \mathbf{R}^4$ be defined by

$$\phi((x, y, z)) = (x + y + 2z, y + z, x + 2y + 3z, x + 3y + 4z).$$

Find a basis for the image of ϕ and a basis for the kernel of ϕ. Find the rank of ϕ. Find the nullity of ϕ. [10 marks]

4. Let A be a point on a line n, let σ_n denote reflection in the line n, and let $\rho_{A,\alpha}$ $(0 \leq \alpha < 2\pi)$ denote rotation anticlockwise about A through angle α. Show that $\rho_{A,\alpha}\sigma_n = \sigma_n\rho_{A,\alpha}$ if and only if $\alpha = 0, \pi$. [10 marks]

5. Let W be the vector space of 2×2 matrices with entries in $\mathbf{R}$. Let U be the subspace of W spanned by

$$u_1 = \begin{pmatrix} 1 & -1 \\ -3 & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 & -1 \\ -5 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.$$

Show that $\left\{\begin{pmatrix} 1 & -1 \\ -3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$ is a basis for U.

Let V be the subspace of W spanned by

$$v_1 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 3 & 2 \\ -4 & 0 \end{pmatrix}.$$

Show that $U = V$. [9 marks]

6. Define a *bilinear form*. Say what it means for a bilinear form to be *symmetric*. Let $\phi : V \times V \to \mathbf{R}$ be defined by

$$\phi((x_1, y_1), (x_2, y_2)) = x_1^2 + x_2^2 + y_1^2 + y_2^2.$$

Show that ϕ is not a bilinear form. [8 marks]

SECTION B

7. Consider the surface $f(x, y, z) = x^2+3y^2+3z^2-2xy+2yz = 5$. Find a linear change of coordinates which diagonalises the quadratic form $f(x, y, z)$. Describe the surface as one of the folowing: Ellipsoid, Hyperboloid of one sheet, Hyperboloid of two sheets, Elliptic Cone. Draw a sketch of the surface. [15 marks]

8. Suppose $\{x_1, \ldots, x_n\}$ is a basis for a vector space V. Describe the dual space V^* to V and describe how to define addition and scalar multiplication on V^* [you need not prove that V^* is a vector space]. Define the dual basis $\{\phi_1, \ldots, \phi_n\}$ to $\{x_1, \ldots, x_n\}$ and prove that it is a basis for V^*.

Consider the basis $\{v_1, v_2\}$ of $\mathbf{R}^2$, where $v_1 = (1, 1)$ and $v_2 = (1, 2)$. Find the dual basis $\{\phi_1, \phi_2\}$ to $\{v_1, v_2\}$. Compute $\phi_1((2, 1))$ and $\phi_2((2, 1))$. [15 marks]

9. Let V be a vector space, and let $f : V \to V$ be a linear map. Define what it means for a $v \in V$ to be an *eigenvector* of f with *eigenvalue* λ.

Let V be the vector space of 2×2 matrices with entries in $\mathbf{R}$. Define the map $f : V \to V$ by

$$f(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A.$$

Describe the eigenvectors and eigenvalues of f. [15 marks]

10. Define what it means for H to be a *subgroup* of a group G. State Lagrange's Theorem.

Let G be the dihedral group D_{2n} [the group of symmetries of a regular polygon of n sides]. Write down a presentation of G [you need not prove that it is a presentation]. Let H be the set of rotational symmetries of a regular polygon of n sides. Show that H is a subgroup of G. Decide whether there exists a group K, distinct from both H and G, such that both H is a subgroup of K, and K is a subgroup of G. [15 marks]