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SECTION A

1.

(a) Let V be a vector space. Say what it means for a subset $\{v_1, \dots, v_k\}$ of V to be *linearly independent*.

(b) Let $V = \mathbb{R}^3$, and let U be the subspace of V spanned by

$$u_1 = (1, -1, 1), \quad u_2 = (1, 2, -1), \quad u_3 = (3, 0, 1).$$

Find the dimension of U .

(c) Let W be the subspace of V spanned by

$$w_1 = (-4, 1, -2), \quad w_2 = (2, 1, 0), \quad w_3 = (5, 1, 1).$$

Show that $U = W$.

[9 marks]

2. Let f be the bilinear form on \mathbb{R}^2 defined by

$$f((x_1, y_1), (x_2, y_2)) = 2x_1x_2 + x_1y_2 + 2y_1y_2.$$

Let $u_1 = (1, -1)$, $u_2 = (1, -2)$. Compute $f(u_1, u_1)$, $f(u_1, u_2)$, $f(u_2, u_1)$, $f(u_2, u_2)$. Find the matrix A of f relative to the basis $\{u_1, u_2\}$. Find the matrix B of f relative to the basis $\{v_1, v_2\}$, where $v_1 = (-2, 1)$, $v_2 = (5, 1)$.

Find the change of basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ and check that $B = P^T A P$.

[9 marks]



THE UNIVERSITY
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3. Let V be the vector space of polynomials in x of degree at most 2 with coefficients in \mathbb{R} . Let the linear map $\varphi : V \rightarrow V$ be defined by

$$\varphi(ax^2 + bx + c) = (3a + b)x^2 + (c + b - 2a)x + 2a + b + c.$$

[You need not show that φ is linear.]

- (a) Find M , the matrix representation of φ with respect to the basis $\{x^2, x, 1\}$.
- (b) What are the eigenvalues and eigenvectors of M ?
- (c) Is M diagonalizable?

[11 marks]

4. Define the terms: *group*, *homomorphism*, *injective*, *surjective*.

Let G be the group of 2×2 matrices with integer entries, under matrix addition. Let H be the group of integers under addition. [You need not show that these are groups]. Let $\varphi : G \rightarrow H$ be defined by

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 3(a + b) - 6(c - d).$$

Show that φ is a homomorphism. State, giving reasons, whether φ is injective. State, giving reasons, whether φ is surjective. [9 marks]

5. Let V and W be finite-dimensional vector spaces, and let $\varphi : V \rightarrow W$ be linear. Define the *rank* and *nullity* of φ . State the rank and nullity theorem.

Let $V = \mathbb{R}^3$, and let $W = \mathbb{R}^{2 \times 2}$ be the space of real 2×2 -matrices. Define $\varphi : V \rightarrow W$ by

$$\varphi(x, y, z) = \begin{pmatrix} x + y + z & z + y \\ 2x - y - z & 0 \end{pmatrix}.$$

Show that φ is a linear map and compute its rank and nullity. [8 marks]

6. Let V be a vector space.

- (a) Say what it means for a function $\varphi : V \rightarrow V$ to be an *isomorphism* of V to itself.
- (b) Show that the set of isomorphisms from V to itself is a group under composition.

[9 marks]



THE UNIVERSITY
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SECTION B

7. Let the linear map $\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be given by

$$\varphi(x, y, z, w) = (x + z - w, 2x - z - 2w, 4x + z - 4w).$$

[You need not show that φ is linear.]

- (a) Find a basis for U , the image of φ , and a basis for W , the kernel of φ . What are the rank and the nullity of φ ?
- (b) Find bases B of \mathbb{R}^4 and C of \mathbb{R}^3 such that the matrix A of φ with respect to B and C is in the standard form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(When you have found B and C , you should compute the matrix A to verify that it has the required form.)

8. Consider the quadratic form

$$q(x, y, z) = 4x^2 - 4y^2 + z^2 + 6xy.$$

Give the matrix A representing q with respect to the standard basis. Find a diagonal matrix D equivalent to A and the matrix P which describes the change of basis from the standard basis to the basis in which q is diagonal. What are the rank and signature of q ? What type of quadric surface is described by the equation $q(x, y, z) = 2$? [15 marks]

9. Let $V = \text{Pol}_3(\mathbb{R})$ be the vector space of polynomials of degree at most three with real coefficients. Let

$$U = \{(2a + b)x^3 + ax^2 - (2a + b)x - b : a, b \in \mathbb{R}\}.$$

Show that U is a subspace of V .

Similarly,

$$W = \{(b - a)x^3 + cx^2 + (a - b)x - c : a, b, c \in \mathbb{R}\}$$

is a subspace of V [you do not need to show this]. What are the dimensions of each of $U, W, U \cap W$ and $U + W$? Is it true or false that $U + W = U \oplus W$?

[15 marks]



THE UNIVERSITY
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10.

(a) Let G be a group. Which of the following statements are true, and which are false? Give counterexamples to any false statements. You do not need to prove the true statements.

(i) Let $a, b \in G$, and suppose that $ab = a$. Then b is the identity element of G .

(ii) Let $a, b \in G$. Then $ab = ba$.

(iii) If $a, b, c \in G$ with $ab = ac$, then $b = c$.

(iv) If $a, b, c \in G$ with $ab = ca$, then $b = c$.

(b) The following is a partially completed group table. Fill in the missing entries. You must justify why each choice of entry is the only one possible.

*	A	B	C	D	E	F
A	B	?	A	E	?	?
B	C	?	?	?	?	?
C	?	?	?	?	?	?
D	?	E	?	C	?	?
E	?	?	?	A	?	?
F	?	?	?	?	?	C

(c) Name a known group with the same group table.

[15 marks]