SECTION A

1. The set $\{u_1, \ldots u_k\}$ spans U if every $u \in U$ can be written as a linear combination $u = \lambda_1 u_1 + \ldots \lambda_k u_k$, for some $\lambda_1, \ldots, \lambda_k \in \mathbf{R}$.

[2 marks]. *Definition from lectures.*

With respect to the standard bais $1, x, x^2$, the vectors are: $u_1 = (1, 2, 5), u_2 = (-1, -1, -1), u_3 = (3, 4, 7)$ First put these as the rows of a matrix, and use row operations to reduce to echelon form:

$$\begin{pmatrix} 1 & 2 & 5 \\ -1 & -1 & -1 \\ 3 & 4 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & -2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the space U is spanned by $\{(1, 0, -3), (0, 1, 4)\}$ which are clearly linearly independent and so give a basis for U.

Similarly put w_1, w_2, w_3 as the rows of a matrix, and use row operations to reduce to echelon form:

$$\begin{pmatrix} 1 & -1 & -7 \\ 1 & 0 & -3 \\ 0 & -1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -7 \\ 0 & 1 & 4 \\ 0 & -1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -7 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the space W also has the same basis as U, namely: $\{(1, 0, -3), (0, 1, 4)\}$, and so $U = W$.

[7 marks]. *Seen similar in exercises.*

9 marks in total for Question 1

2. A *group* is a set G together with a binary operation $*$ such that: **(1)** for all $g_1, g_2 \in G$, $g_1 * g_2 \in G$; **(2)** for all $g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$; **(3)** there exists an element $e \in G$ such that, for all $g \in G$, $e * g = g * e = g$; **(4)** for every $g \in G$, there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$. If G, H are groups, then a map $\phi : G \to H$ is a *homomorphism* if, for all $g_1, g_2 \in G$, $\phi(g_1 *_1 g_2) = \phi(g_1) *_2 \phi(g_2)$, where $*_1$ is the group law in G and $*_2$ is the group law in H. The map ϕ is *injective* if, for all $g_1, g_2 \in G$, $\phi(g_1) = \phi(g_2) \Rightarrow g_1 = g_2$. The map ϕ is *surjective* if, for all $h \in H$, there exists $g \in G$ such that $\phi(g) = h$.

[5 marks]. *Standard definitions from lectures.*

For any $g_1, g_2 \in G$, $\phi(g_1 + g_2) = \left(\begin{smallmatrix} 1 & 0 \\ 3(g_1+g_2) & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 3g_1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 3g_2 & 1 \end{smallmatrix}\right) = \phi(g_1)\phi(g_2)$. Hence ϕ is a homomorphism.

For any $g_1, g_2 \in G$, $\phi(g_1) = \phi(g_2) \Rightarrow \left(\begin{smallmatrix} 1 & 0 \\ 3g_1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 3g_2 & 1 \end{smallmatrix}\right) \Rightarrow 3g_1 = 3g_2 \Rightarrow g_1 = g_2$, so that ϕ is injective.

The element $\left(\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right) \in H$, for example, does not occur as $\phi(g)$ for any $g \in G$ (since $\phi(g)$ always has 1 as its top left hand entry), so that ϕ is not surjective.

[4 marks]. *Seen somewhat similar in exercises.*

9 marks in total for Question 2

3. Let $e_1 = 1, e_2 = x, e_3 = x^2, e_4 = x^3$. Then $L(e_1) = L(1) = 0 = 0 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 + 0 \cdot e_4$, so that the first column of the matrix should have entries $0, 0, 0, 0$. Similarly, $L(e_2) = 0 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 + 1 \cdot e_4$, $L(e_3) = 0 \cdot e_1 + 0 \cdot e_2 + (-1) \cdot e_3 + 0 \cdot e_4$, and $L(e_4) = 0 \cdot e_1 + 1 \cdot e_2 + 0 \cdot e_3 + 0 \cdot e_4$, so that the matrix is: $M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

[3 marks].

We now compute $\det(\lambda I - M)$, using first $r_2 \leftrightarrow r_4$ (which negates the determinant) and then $r_4 \to r_4 + \lambda r_2$ (which leave the determinant unchanged), as follows:

$$\det \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda+1 & 0 \\ 0 & -1 & 0 & \lambda \end{pmatrix} = -\det \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -1 & 0 & \lambda \\ 0 & 0 & \lambda+1 & 0 \\ 0 & \lambda & 0 & -1 \end{pmatrix}$$

$$= -\det \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -1 & 0 & \lambda \\ 0 & 0 & \lambda+1 & 0 \\ 0 & 0 & 0 & \lambda^2 - 1 \end{pmatrix} = -\lambda(-1)(\lambda+1)(\lambda^2-1),$$

which is $\lambda(\lambda - 1)(\lambda + 1)^2$. We therefore see that the possible eigenvalues are $\lambda = 0, 1, -1$.

When $\lambda = 0$, a vector $v = a + bx + cx^2 + dx^3$ is an eigenvector with eigenvalue 0 iff $L(v) = \lambda v$ iff $dx - cx^2 + bx^3 = 0$ iff $b = 0, c = 0, d = 0$ iff v is of the form a $(a \neq 0)$.

When $\lambda = 1$, a vector $v = a + bx + cx^2 + dx^3$ is an eigenvector with eigenvalue 1 iff $L(v) = \lambda v$ iff $dx - cx^2 + bx^3 = a + bx + cx^2 + dx^3$ iff $0 = a, d = b, -c = c, b = d$ iff $b = d$ and $a = c = 0$ iff v is of the form $bx + bx^3$ $(b \neq 0)$.

When $\lambda = -1$, a vector $v = a + bx + cx^2 + dx^3$ is an eigenvector with eigenvalue -1 iff $L(v) = \lambda v$ iff $dx - cx^2 + bx^3 = -a - bx - cx^2 - dx^3$ iff $0 = -a, d = -b, -c = -c, b = -d$ iff v is of the form $bx + cx^2 - bx^3$ (b, c not both 0).

[6 marks]. *Whole question: seen similar (once) in exercises.*

9 marks in total for Question 3

4. The *rank* of F is the dimension of im F (where im F = the image of $F = \{F(v) : v \in V\}$). The *nullity* of F is the dimension of ker F (where ker F = the kernel of $F = \{v \in V : F(v) = 0\}$). The rank & nullity theorem states that the rank of F plus the nullity of F is $\dim(V)$.

[2 marks]

Applying column operations to the matrix for F gives:

$$\begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & -3 & 0 & 3 \\ 1 & 0 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 2 \\ 1 & 2 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}.$$

A basis for the image of F is given by the linearly independent columns, namely: $\{(1,1,1),(0,-1,2)\}$. The image of F therefore has dimension 2 and so the rank is 2.

[3 marks]

Applying row operations to the matrix for F gives:

$$\begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & -3 & 0 & 3 \\ 1 & 0 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & -1 & -1 & 2 \\ 0 & 2 & 2 & -4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & -3 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, (x,y,s,t) is in the kernel of F iff it satisfies $F((x,y,s,t)) = (0,0,0)$; that is to say:

$$\begin{pmatrix} 1 & 0 & 3 & -3 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The general solution is:

$$(x,y,s,t) = (-3s+3t, -s+2t, s, t) = s(-3,-1,1,0) + t(3,2,0,1).$$

A basis is therefore $\{(-3,-1,1,0),(3,2,0,1)\}$ [which are easily checked as being linearly independent], and so the dimension of the kernel is 2, giving that the nullity is 2.

[4 marks]

We now observe that rank + nullity = 4, which is indeed the dimension of V.

[1 mark] *Whole question: seen similar in exercises.*

10 marks in total for Question 4

5. We compute:

$f(u_1, u_1) = 2 \cdot 2 \cdot 2 - 0 \cdot 2 + 0 \cdot 0 = 8,$

$f(u_1, u_2) = 2 \cdot 2 \cdot (-1) - 0 \cdot (-1) + 0 \cdot 3 = -4,$

$f(u_2, u_1) = 2 \cdot (-1) \cdot 2 - 3 \cdot 2 + 3 \cdot 0 = -10,$

$f(u_2, u_2) = 2 \cdot (-1) \cdot (-1) - 3 \cdot (-1) + 3 \cdot 3 = 14.$

So, the matrix of f wrt u_1, u_2 is $A = \left(\begin{smallmatrix} 8 & -4 \\ -10 & 14 \end{smallmatrix}\right)$.

[3 marks]

Similarly, $f(v_1, v_1) = 2 \cdot 1 \cdot 1 - 3 \cdot 1 + 3 \cdot 3 = 8$, $f(v_1, v_2) = 2 \cdot 1 \cdot 3 - 3 \cdot 3 + 3 \cdot 3 = 6$, $f(v_2, v_1) = 2 \cdot 3 \cdot 1 - 3 \cdot 1 + 3 \cdot 3 = 12$, $f(v_2, v_2) = 2 \cdot 3 \cdot 3 - 3 \cdot 3 + 3 \cdot 3 = 18$. So, the matrix of f wrt v_1, v_2 is $B = \left(\begin{smallmatrix} 8 & 6 \\ 12 & 18 \end{smallmatrix}\right)$.

[3 marks]

Now, note that $v_1 = 1 \cdot u_1 + 1 \cdot u_2$, so that "1" and "1" are the entries of the first column of the change-of-basis matrix. Similarly, $v_2 = 2 \cdot u_1 + 1 \cdot u_2$, so that "2" and "1" are the entries of the second column of the change-of-basis matrix. This gives $P = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$ as the required change-of-basis matrix. Finally, check that: $P^T A P = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)^T \left(\begin{smallmatrix} 8 & -4 \\ -10 & 14 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 8 & -4 \\ -10 & 14 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 4 & 12 \\ 4 & -6 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 8 & 6 \\ 12 & 18 \end{smallmatrix}\right) = B$, as required.

[3 marks]. *Whole question: seen similar (once) in exercises.*

9 marks in total for Question 5

6. A matrix M is *orthogonal* if $MM^T = I$ or, equivalently, $M^T M = I$.

[2 marks]. *Standard.*

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We are given that $\|Mv\| = \|v\|$ for all $v \in \mathbf{R}^2$ so, in particular $\|M\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\|^2 = \|\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\|^2 = 1$, $\|M\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\|^2 = \|\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\|^2 = 1$ and $\|M\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\|^2 = \|\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\|^2 = 2$. This gives the three equations: $a^2 + c^2 = 1, b^2 + d^2 = 1$ and $(a+b)^2 + (c+d)^2 = 2$. Subtracting the first two equations from the third gives: $2ab + 2cd = 0$ and so $ab + cd = 0$. Hence $M^T M = \left(\begin{smallmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, giving that M is orthogonal, as required.

[7 marks]. *Unseen, but with a substantial hint.*

9 marks in total for Question 6

SECTION B

7. A typical member of U is $\begin{pmatrix} a & b \\ a+b & d \end{pmatrix}$. Then $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in U$ by taking $a = b = d = 0$. If $\begin{pmatrix} a & b \\ a+b & d \end{pmatrix} \in U$ and $\lambda \in \mathbf{R}$ then $\lambda\begin{pmatrix} a & b \\ a+b & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda(a+b) & \lambda d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda a+\lambda b & \lambda d \end{pmatrix} \in U$. Finally, if $\begin{pmatrix} a_1 & b_1 \\ a_1+b_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ a_2+b_2 & d_2 \end{pmatrix} \in U$ then $\begin{pmatrix} a_1 & b_1 \\ a_1+b_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ a_2+b_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ (a_1+b_1)+(a_2+b_2) & d_1+d_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ (a_1+a_2)+(b_1+b_2) & d_1+d_2 \end{pmatrix} \in U$. Hence, U is a subspace.

A typical member of W is $\begin{pmatrix} c & c \\ c & c \end{pmatrix}$. Then $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ by taking $c = 0$. If $\begin{pmatrix} c & c \\ c & c \end{pmatrix} \in W$ and $\lambda \in \mathbf{R}$ then $\lambda\begin{pmatrix} c & c \\ c & c \end{pmatrix} = \begin{pmatrix} \lambda c & \lambda c \\ \lambda c & \lambda c \end{pmatrix} \in W$. Finally, if $\begin{pmatrix} c_1 & c_1 \\ c_1 & c_1 \end{pmatrix}, \begin{pmatrix} c_2 & c_2 \\ c_2 & c_2 \end{pmatrix} \in W$ then $\begin{pmatrix} c_1 & c_1 \\ c_1 & c_1 \end{pmatrix} + \begin{pmatrix} c_2 & c_2 \\ c_2 & c_2 \end{pmatrix} = \begin{pmatrix} c_1+c_2 & c_1+c_2 \\ c_1+c_2 & c_1+c_2 \end{pmatrix} \in W$. Hence W is a subspace.

[4 marks].

Consider $M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $M_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then any member of U, $\begin{pmatrix} a & b \\ a+b & d \end{pmatrix}$, can be written as a linear combination: $aM_1 + bM_2 + dM_3$, and so M_1, M_2, M_3 span U. Also, if $\lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3 = 0$, then $\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_1+\lambda_2 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and so $\lambda_1 = \lambda_2 = \lambda_3 = 0$, giving that M_1, M_2, M_3 are linearly independent. We conclude that $\{M_1, M_2, M_3\}$ is a basis for U and so U has dimension 3.

[Merely stating without justification that $\dim(U) = 3$ gets 1 mark; similarly for each of $\dim(W), \dim(U \cap W), \dim(U + W)$, below]

[3 marks].

A typical member of W looks like: $\begin{pmatrix} c & c \\ c & c \end{pmatrix}$. Clearly, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is a basis and so W has dimension 1.

[2 marks].

A matrix A is in $U \cap W$ if A is of the form $\begin{pmatrix} a & b \\ a+b & d \end{pmatrix}$ and $\begin{pmatrix} c & c \\ c & c \end{pmatrix}$ simultaneously, so that $a = c, b = c, a + b = c, d = c$; the first plus the second minus the third equation gives $c = 0$, and $a = b = d = 0$ also, which is only possible when $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; that is, $U \cap W$ consists only of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and so has dimension 0.

[2 marks].

Any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $M_2(\mathbf{R})$ can be written as: $\begin{pmatrix} c-b & c-a \\ 2c-a-b & c+d-a-b \end{pmatrix} + \begin{pmatrix} a+b-c & a+b-c \\ a+b-c & a+b-c \end{pmatrix}$, where the first addend is in U and the second in W. Therefore, anything in $M_2(\mathbf{R})$ can be written as an element of U plus an element of W, and so: $U + W = M_2(\mathbf{R})$, which has dimension 4 [since, for example, the standard basis: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ has size 4]. [It is also acceptible to state and apply: $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$.]

[3 marks].

Finally, it is true that $V = U \oplus W$, since we have both $V = U + W$ and $U \cap W = \{0\}$.

[1 mark].

15 marks in total for Question 7. *Unseen.*

8. The dual space V^* is defined to be the set of all linear maps from V to $\mathbf{R}$. Given $\theta, \phi \in V^*$, we can define $\theta + \phi$ by: $(\theta + \phi)(x) = \theta(x) + \phi(x)$, for all $x \in V$. Similarly, for $\lambda \in \mathbf{R}$, define $\lambda\theta$ by $(\lambda\theta)(x) = \lambda(\theta(x))$, for all $x \in V$. Given a basis $\{x_1, \ldots, x_n\}$ for V, the i-th member of the dual basis, ϕ_i, is defined to be the unique linear map from V to $\mathbf{R}$ such that $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$, for all $j \neq i$. Suppose $f \in V^*$; define $\lambda_j = f(x_j)$ for all j; then $(\lambda_1\phi_1 + \ldots + \lambda_n\phi_n)(x_j) = \lambda_j \cdot \phi_j(x_j)$ [since $\phi_i(x_j) = 0$, for all $j \neq i$] $= \lambda_j$ [since $\phi_j(x_j) = 1$]. Hence, f and $\lambda_1\phi_1 + \ldots + \lambda_n\phi_n$ both take the same values on each of $x_1, \ldots x_n$, giving that $f = \lambda_1\phi_1 + \ldots + \lambda_n\phi_n$ [since any linear map is completely determined by its values on a basis]. Hence, $\{\phi_1, \ldots \phi_n\}$ spans V^*. Now suppose that $\lambda_1\phi_1 + \ldots + \lambda_n\phi_n = 0$ for some $\lambda_1, \ldots, \lambda_n$. Then, for any j, $(\lambda_1\phi_1 + \ldots + \lambda_n\phi_n)(x_j) = 0$, and so $\lambda_j \cdot 1 = 0$; hence $\lambda_1 = \ldots = \lambda_n = 0$, and so $\phi_1, \ldots \phi_n$ are linearly independent. Hence $\{\phi_1, \ldots \phi_n\}$ is a basis for V^*.

[**8 marks**] *Bookwork*

In the given example, $\phi_1((x, y, z)) = a_1x + b_1y + c_1z \in V^*$ is defined to satisfy $\phi_1(v_1) = 1$, $\phi_1(v_2) = 0$, $\phi_1(v_3) = 0$; similarly, $\phi_2((x, y, z)) = a_2x + b_2y + c_2z \in V^*$ is defined to satisfy $\phi_2(v_1) = 0$, $\phi_2(v_2) = 1$, $\phi_2(v_3) = 0$; similarly, $\phi_3((x, y, z)) = a_3x + b_3y + c_3z \in V^*$ is defined to satisfy $\phi_3(v_1) = 0$, $\phi_3(v_2) = 0$, $\phi_3(v_3) = 1$;

[**2 marks**]

That is: $2a_1 + b_1 - 2c_1 = 1, b_1 - c_1 = 0, a_1 - 2b_1 + 4c_1 = 0$, which has solution: $a_1 = 2/5, b_1 = -1/5, c_1 = -1/5$, so that ϕ_1 is defined by: $\phi_1((x, y, z)) = (2/5)x - (1/5)y - (1/5)z$. Similarly, $2a_2 + b_2 - 2c_2 = 0, b_2 - c_2 = 1, a_2 - 2b_2 + 4c_2 = 0$, which has solution: $a_2 = 0, b_2 = 2, c_2 = 1$, so that ϕ_2 is defined by: $\phi_2((x, y, z)) = 2y + z$. Similarly, $2a_3 + b_3 - 2c_3 = 0, b_3 - c_3 = 0, a_3 - 2b_3 + 4c_3 = 1$, which has solution: $a_3 = 1/5, b_3 = 2/5, c_3 = 2/5$, so that ϕ_3 is defined by: $\phi_3((x, y, z)) = (1/5)x + (2/5)y + (2/5)z$. Hence, $\phi_1((3, 2, 1)) = 6/5 - 2/5 - 1/5 = 3/5$, $\phi_2((3, 2, 1)) = 0 + 4 + 1 = 5$ and $\phi_3((3, 2, 1)) = 3/5 + 4/5 + 2/5 = 9/5$.

[**5 marks**] *Seen similar in exercises*

15 marks in total for Question 8

9. Taking the standard matrix A for q, we form $(A|I)$. We then apply to A: $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 + 2R_1$, $C_2 \to C_2 - 3C_1$, $C_3 \to C_3 + 2C_1$ as step one and $R_3 \to R_3 + R_2$, $C_3 \to C_3 + C_2$ as step two (with only the column operations being applied to I) to give:

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 3 & 4 & -1 & 0 & 1 & 0 \\ -2 & -1 & -3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & -5 & 5 & 0 & 1 & 0 \\ 0 & 5 & -7 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & -1 \\ 0 & -5 & 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 0 & 0 & 1 \end{array}\right) = (D|P).$$

[7 marks].

Then $D = P^T AP$, and $A = Q^T DQ$, where $Q = P^{-1} = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$. New variables: $\begin{pmatrix} r \\ s \\ t \end{pmatrix} = Q \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (that is, we are changing to new variables r, s, t, where $r = x+3y-2z$, $s = y - z$, $t = z$) transform $q(x, y, z)$ into $\tilde{q}(r, s, t) = r^2 - 5s^2 - 2t^2$.

[3 marks]

The rank of q is 3 (which is the number of nonzero entries of D), and the signature of q is the number of positive entries of D minus the number of negative entries $= 1 - 2 = -1$. The surface $q(x, y, z) = 7$ becomes $r^2 - 5s^2 - 2t^2 = 7$, in r, s, t coordinates, which is a hyperboloid of two sheets. The sketch should look identical to the standard sketch of a hyperboloid of two sheets, except that the x, y, z axes should be labelled r, s, t (if drawn it wrt r, s, t). [If drawn wrt x, y, z then it should be made clear in the diagram that the axes of the surface are: $y = z = 0$, $x + 3y = z = 0$, $x + z = y - z = 0$].

[5 marks]. *Whole question: seen similar in exercises.*

15 marks in total for Question 9

10. (i) Suppose e_1, e_2 are both (2-sided) identity elements. Then $e_1 * e_2 = e_1$, since e_2 is an identity. Similarly, $e_1 * e_2 = e_2$. Hence $e_1 = e_2$. **[2 marks].**

Let $\alpha * \beta = e$. Let δ be the (2-sided) inverse of α, and multiply both sides of the equation on the left by δ. Then $\delta * (\alpha * \beta) = \delta * e = \delta$ (since e is identity), so that $(\delta * \alpha) * \beta = \delta$ (assoc.) and so $\beta = \delta$. Now multiply both sides on the right by α, giving $\beta * \alpha = \delta * \alpha = e$. **[2 marks].** *Unseen*

(ii) Suppose $\alpha * \beta = \alpha * \gamma$. Multiply both sides on the left by δ, the inverse of α. Then $\delta * (\alpha * \beta) = \delta * (\alpha * \gamma)$, giving $(\delta * \alpha) * \beta = (\delta * \alpha) * \gamma$ [by associativity], and so $e * \beta = e * \gamma$, finally giving: $\beta = \gamma$, as required. The values of $\alpha * g$, as g runs through all the members of the group give the 'α' row of the group table; if two of these were the same, we would have $\alpha * \beta = \alpha * \gamma$ for distinct $\beta \neq \gamma$, contradicting the previous result. Similarly, $\beta * \alpha = \gamma * \alpha \Rightarrow \beta = \gamma$ gives that no element can be repeated in the same column. **[4 marks].** *Seen on ex sheet.*

(iii) From the already provided entry $E * D = D$, we deduce (after multiplying both sides on the right by the inverse of D) that E is the identity element. This allows us to fill in the bottom row as ABCDE and similarly the right hand column. We are given that $D * A = E$, the identity so, by part (i), $A * D = E$.

*	A	B	C	D	E
A	?	?	?	E	A
B	?	?	?	?	?
C	?	?	?	?	?
D	E	?	B	?	?
E	A	B	C	D	E

Now fill in all remaining entries using the 'no-element-repeated-in-the-same-row-or-column' rule, with the following possible order.

*	A	B	C	D	E
A	3	2	1	E	A
B	5	8	10	11	B
C	4	7	9	12	C
D	E	6	B	13	D
E	A	B	C	D	E

The final table must then be

*	A	B	C	D	E
A	B	C	D	E	A
B	C	D	E	A	B
C	D	E	A	B	C
D	E	A	B	C	D
E	A	B	C	D	E

[7 marks]. *Seen similar on Ex Sheet.* **15 marks in total for Question 10**