SECTION A

1. Say what it means for $\{u_1, \ldots, u_k\}$ to *span* a vector space U.

Let V be the vector space of polynomials of degree at most 2 with coefficients in $\mathbf{R}$. Let U be the subspace of V spanned by

$$u_1 = 1 + 2x + 5x^2,\ u_2 = -1 - x - x^2,\ u_3 = 3 + 4x + 7x^2.$$

Let W be the subspace of V spanned by

$$w_1 = 1 - x - 7x^2,\ w_2 = 1 - 3x^2,\ w_3 = -x - 4x^2.$$

Show that $U = W$. [9 marks]

2. Define the terms: *group, homomorphism, injective, surjective.*

Let G be the group of real numbers under addition; let H be the group of 2×2 matrices with real entries, nonzero determinant, and top right hand entry equal to zero, under the operation of matrix multiplication [you need not show that these are groups]. Let $\phi : G \to H$ be defined by

$$\phi(g) = \begin{pmatrix} 1 & 0 \\ 3g & 1 \end{pmatrix}.$$

Show that ϕ is a homomorphism. State, giving reasons, whether ϕ is injective. State, giving reasons, whether ϕ is surjective. [9 marks]

3. Let V be the vector space of polynomials in x of degree at most 3 with coefficients in $\mathbf{R}$. Let the linear map $L : V \to V$ be defined by

$$L(a + bx + cx^2 + dx^3) = dx - cx^2 + bx^3.$$

Find M, the matrix representation of L with respect to the basis $\{1, x, x^2, x^3\}$. What are the eigenvalues and eigenvectors of L? [9 marks]

4. Define the *rank* and *nullity* of a linear map $F : V \to W$. State the rank & nullity theorem.

Let $F : \mathbf{R}^4 \to \mathbf{R}^3$ be the linear mapping defined by:

$$F((x, y, s, t)) = (x - 2y + s + t, x - 3y + 3t, x + 3s - 3t).$$

Find a basis for the image of F, and hence the rank of F. Find a basis for the kernel of F, and hence the nullity of F. Verify that the rank & nullity theorem holds in this case. [10 marks]

5. Let f be the bilinear form on $\mathbf{R}^2$ defined by

$$f((x_1, x_2), (y_1, y_2)) = 2x_1y_1 - x_2y_1 + x_2y_2.$$

Let $u_1 = (2, 0), u_2 = (-1, 3)$. Compute $f(u_1, u_1), f(u_1, u_2), f(u_2, u_1), f(u_2, u_2)$. Find the matrix A of f relative to the basis $\{u_1, u_2\}$. Find the matrix B of f relative to the basis $\{v_1, v_2\}$, where $v_1 = (1, 3), v_2 = (3, 3)$.

Find the change of basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ and show that $B = P^T AP$. [9 marks]

6. Define what it means for a matrix to be *orthogonal.*

Let M be a 2×2 matrix with the property that $\|Mv\| = \|v\|$ for all v in $\mathbf{R}^2$. Show that M is an orthogonal matrix. [Hint: Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and compute $M^T M$. Consider each of $\|M\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\|$, $\|M\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\|$ and $\|M\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\|$.] [9 marks]

SECTION B

7. Let V be the vector space of 2×2 matrices with entries in $\mathbf{R}$. Let

$$U = \{\begin{pmatrix} a & b \\ a+b & d \end{pmatrix} : a, b, d \in \mathbf{R}\},$$

and

$$W = \{\begin{pmatrix} c & c \\ c & c \end{pmatrix} : c \in \mathbf{R}\}.$$

Show that U and W are subspaces of V. What are the dimensions of each of $U, W, U \cap W$ and $U + W$? Is it true or false that $V = U \oplus W$? [15 marks]

8. Suppose $\{x_1, \dots, x_n\}$ is a basis for a vector space V. Describe the dual space V^* to V and describe how to define addition and scalar multiplication on V^* [you need not prove that V^* is a vector space]. Define the dual basis $\{\phi_1, \dots, \phi_n\}$ to $\{x_1, \dots, x_n\}$ and prove that it is a basis for V^*.

Consider the basis $\{v_1, v_2, v_3\}$ of $\mathbf{R}^3$, where $v_1 = (2, 1, -2)$, $v_2 = (0, 1, -1)$ and $v_3 = (1, -2, 4)$. Find the dual basis $\{\phi_1, \phi_2, \phi_3\}$ to $\{v_1, v_2, v_3\}$. Compute $\phi_1((3, 2, 1))$, $\phi_2((3, 2, 1))$ and $\phi_3((3, 2, 1))$. [15 marks]

9. Consider the quadratic form

$$q(x, y, z) = x^2 + 6xy - 4xz + 4y^2 - 2yz - 3z^2.$$

Give the matrix A representing q with respect to the standard basis. Find a diagonal matrix D equivalent to A and the matrix P which describes the change of basis from the standard basis to the basis in which q is diagonal. Find the change of variables corresponding to this change of basis. What are the rank and signature of q? Describe geometrically the surface $q(x, y, z) = 7$. Draw a sketch of the surface. [15 marks]

10.(i) Let G be a group. Show that the identity element e is unique. Show that $\alpha * \beta = e \Rightarrow \beta * \alpha = e$, for any $\alpha, \beta \in G$.

(ii) Show that, for any $\alpha, \beta, \gamma \in G$, $\alpha * \beta = \alpha * \gamma \Rightarrow \beta = \gamma$. Deduce that no element can be repeated in the same row inside a group table. Similarly show that no element can be repeated in the same column.

(iii) The following is a partially completed group table for a group with five elements. Fill in the missing entries. You must justify (entry by entry) why each choice of entry is the only one possible.

*	A	B	C	D	E
A	?	?	?	?	?
B	?	?	?	?	?
C	?	?	?	?	?
D	E	?	B	?	?
E	?	?	?	D	?

[15 marks]