SECTION A

1. A finite set of vectors $S = \{v_1, \ldots v_n\}$ is a *basis* for V if: **(1)** S spans V – that is, every $v \in V$ can be written as a finite linear combination of members of S; **(2)** S is linearly independent – that is, whenever $\lambda_1 v_1 + \ldots \lambda_n v_n = 0$ then $\lambda_1 = \ldots = \lambda_n = 0$

[2 marks]. *Definition from lectures.*

For the set $\{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$, if we write the vectors wrt the standard basis $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ they are: $(0,1,1,1), (1,0,1,1), (1,1,0,1), (1,1,1,0)$. Putting these as the rows of a 4×4 matrix, we can use a few elementary row operations (namely: $r_1 \leftrightarrow r_2$, $r_3 \to r_3 - r_1$, $r_4 \to r_4 - r_1$, $r_3 \to r_3 - r_2$, $r_4 \to r_4 - r_2$, $r_3 \to (-1/2)r_3$, $r_1 \to r_1 - r_3$, $r_2 \to r_2 - r_3$, $r_4 \to r_4 + r_3$, $r_4 \to (-2/3)r_4$, $r_1 \to r_1 - (1/2)r_4$, $r_2 \to r_2 - (1/2)r_4$, $r_3 \to r_3 - (1/2)r_4$) to obtain the identity matrix, so that the given set is a basis [or, show directly from definitions that the set spans V and is linearly independent].

[4 marks]. *Seen similar in lectures.*

The set $\{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \}$ is not linearly independent, since:
$1 \cdot \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) + (-1) \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) + 1 \cdot \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right) + (-1)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and so the set is not a basis.

[2 marks]. *Seen similar in exercises.*

8 marks in total for Question 1

2. A *group* is a set G together with a binary operation $*$ such that: **(1)** for all $g_1, g_2 \in G$, $g_1 * g_2 \in G$; **(2)** for all $g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$; **(3)** there exists an element $e \in G$ such that, for all $g \in G$, $e * g = g * e = g$; **(4)** for every $g \in G$, there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$. If G, H are groups, then a map $\phi : G \to H$ is a *homomorphism* if, for all $g_1, g_2 \in G$, $\phi(g_1 *_1 g_2) = \phi(g_1) *_2 \phi(g_2)$, where $*_1$ is the group law in G and $*_2$ is the group law in H. The map ϕ is *injective* if, for all $g_1, g_2 \in G$, $\phi(g_1) = \phi(g_2) \Rightarrow g_1 = g_2$. The map ϕ is *surjective* if, for all $h \in H$, there exists $g \in G$ such that $\phi(g) = h$.

[5 marks]. *Standard definitions from lectures.*

For any $g_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ 0 & d_1 \end{smallmatrix}\right), g_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ 0 & d_2 \end{smallmatrix}\right) \in G$ we have
$\phi(g_1 g_2) = \phi(\left(\begin{smallmatrix} a_1 & b_1 \\ 0 & d_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_2 & b_2 \\ 0 & d_2 \end{smallmatrix}\right)) = \phi(\left(\begin{smallmatrix} a_1 a_2 & a_1 b_2 + b_1 d_2 \\ 0 & d_1 d_2 \end{smallmatrix}\right)) = (a_2 a_2)^2 = a_1^2 a_2^2 = \phi(g_1)\phi(g_2)$.
Hence ϕ is a homomorphism.

$\phi(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = 1 = \phi(\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right))$, for example, so that ϕ is not injective.

For any $h \in H$, we have that h is a positive real number, and so:
$\phi(\left(\begin{smallmatrix} \sqrt{h} & 0 \\ 0 & 1 \end{smallmatrix}\right)) = (\sqrt{h})^2 = h$. Hence ϕ is surjective.

[4 marks]. *Seen somewhat similar in exercises.*

9 marks in total for Question 2

3. The *rank* of F is the dimension of im F (where im F = the image of $F = \{F(v) : v \in V\}$). The *kernel* of F is the dimension of ker F (where ker F = the kernel of $F = \{v \in V : F(v) = 0\}$). The rank & nullity theorem states that the rank of F plus the nullity of F is $\dim(V)$.

[2 marks]

Applying column operations to the matrix for F gives:

$$\begin{pmatrix} 1 & 3 & 0 & 2 \\ -1 & -2 & 1 & 0 \\ 2 & 3 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 2 \\ 2 & -3 & 0 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 \end{pmatrix}.$$

A basis for the image of F is given by the linearly independent columns, namely: $\{(1,-1,2),(0,1,-3),(0,0,3)\}$. The image of F therefore has dimension 3 and so the rank is 3.

[3 marks]

Applying row operations to the matrix for F gives:

$$\begin{pmatrix} 1 & 3 & 0 & 2 \\ -1 & -2 & 1 & 0 \\ 2 & 3 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & 0 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

So, (x,y,s,t) is in the kernel of F iff it satisfies $F((x,y,s,t)) = (0,0,0)$; that is to say:

$$\begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The general solution is: $(x,y,s,t) = (4t,-2t,0,t) = t(4,-2,0,1)$. A basis is therefore $\{(4,-2,0,1)\}$ [which contains a single nonzero vector and so is clearly linearly independent], and so the dimension of the kernel is 1, giving that the nullity is 1.

[4 marks]

We now observe that rank + nullity = 4, which is indeed the dimension of V.

[1 mark] *Whole question: seen similar in exercises.*

10 marks in total for Question 3

4. (i) First note that $\sigma_\ell, \sigma_m, \rho_{A,2\alpha}$ all leave A unchanged, so that $\sigma_m\sigma_\ell(A) = A = \rho_{A,2\alpha}(A)$. Now, let B be any point on ℓ distinct from A, let $B' = \sigma_m(B)$ and let n be the line through A and B'. Let the point Q be the intersection of m and the line BB'. Now, $|AQ| = |AQ|$ and $|BQ| = |B'Q|$ and angle AQB equals angle AQB' equals $\pi/2$. So, triangle AQB is congruent to AQB', giving that $|AB| = |AB'|$ and angle QAB' is the same as angle BAQ, namely: α. It follows that $B' = \rho_{A,2\alpha}(B)$. Further, $\sigma_\ell(B) = B$, since B lies on ℓ. So, we've shown that $\sigma_m\sigma_\ell(B) = B' = \rho_{A,2\alpha}(B)$. Similarly, let k be the line through A at angle $-\alpha$ from ℓ, and let C be any point on k distinct from A. By a simlar argument to above, $\sigma_m\sigma_\ell(C) = \rho_{A,2\alpha}(C)$. This shows that $\sigma_m\sigma_\ell$ and $\rho_{A,2\alpha}$ agree on the three non-collinear points A, B, C. Since these are isometries, and since any isometry is determined by its effect on 3 non-collinear points, we conclude that $\sigma_m\sigma_\ell = \rho_{A,2\alpha}$, as required [it helps also to draw a quick diagram of the above].

[**5 marks**]. *Bookwork from lectures.*

(ii) Let r be the line through B at angle $-\beta/2$ from s. By part (i), we have: $\sigma_s\sigma_r = \rho_{B,2(\beta/2)} = \rho_{B,\beta}$. Similarly, let t be the line through B at angle $\beta/2$ from s. By part (i), we have: $\sigma_t\sigma_s = \rho_{B,2(\beta/2)} = \rho_{B,\beta}$. So, $\rho_{B,\beta}\sigma_s = \sigma_s\rho_{B,\beta} \iff (\sigma_t\sigma_s)\sigma_s = \sigma_s(\sigma_s\sigma_r) \iff \sigma_t(\sigma_s\sigma_s) = (\sigma_s\sigma_s)\sigma_r \iff \sigma_t = \sigma_r \iff t = r \iff$ the angle between r and t is 0 or $\pi \iff \beta/2 + \beta/2 = 0$ or π [since the angle from r to t is the "angle from r to s plus angle from s to t] $\iff \beta = 0$ or π, as required.

[**5 marks**]. *Seen similar in exercises.*

10 marks in total for Question 4

5. We compute: $f(u_1,u_1) = 1\cdot 1 - 2\cdot 1\cdot 2 + 2\cdot 2 = 1$, $f(u_1,u_2) = 1\cdot(-2) - 2\cdot 1\cdot 3 + 2\cdot 3 = -2$, $f(u_2,u_1) = (-2)\cdot 1 - 2\cdot(-2)\cdot 2 + 3\cdot 2 = 12$, $f(u_2,u_2) = (-2)\cdot(-2) - 2\cdot(-2)\cdot 3 + 3\cdot 3 = 25$. So, the matrix of f wrt u_1, u_2 is $A = \left(\begin{smallmatrix} 1 & -2 \\ 12 & 25 \end{smallmatrix}\right)$.

[3 marks]

Similarly, $f(v_1,v_1) = (-1)\cdot(-1) - 2\cdot(-1)\cdot 5 + 5\cdot 5 = 36$, $f(v_1,v_2) = (-1)\cdot 4 - 2\cdot(-1)\cdot 1 + 5\cdot 1 = 3$, $f(v_2,v_1) = 4\cdot(-1) - 2\cdot 4\cdot 5 + 1\cdot 5 = -39$, $f(v_2,v_2) = 4\cdot 4 - 2\cdot 4\cdot 1 + 1\cdot 1 = 9$. So, the matrix of f wrt u_1, u_2 is $B = \left(\begin{smallmatrix} 36 & 3 \\ -39 & 9 \end{smallmatrix}\right)$.

[3 marks]

Now, note that $v_1 = 1\cdot u_1 + 1\cdot u_2$, so that "1" and "1" are the entries of the first column of the change-of-basis matrix. Similarly, $v_2 = 2\cdot u_1 + (-1)\cdot u_2$, so that "2" and "-1" are the entries of the second column of the change-of-basis matrix. This gives $P = \left(\begin{smallmatrix} 1 & 2 \\ 1 & -1 \end{smallmatrix}\right)$ as the required change-of-basis matrix. Finally, check that: $P^T A P = \left(\begin{smallmatrix} 1 & 2 \\ 1 & -1 \end{smallmatrix}\right)^T \left(\begin{smallmatrix} 1 & -2 \\ 12 & 25 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 2 \\ 1 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 2 & -1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -2 \\ 12 & 25 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 2 \\ 1 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 2 & -1 \end{smallmatrix}\right) \left(\begin{smallmatrix} -1 & 4 \\ 37 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 36 & 3 \\ -39 & 9 \end{smallmatrix}\right) = B$, as required.

[3 marks]. *Whole question: seen similar (once) in exercises.*

9 marks in total for Question 5

6. Let $e_1 = 1, e_2 = x, e_3 = x^2, e_4 = x^3$. Then $L(e_1) = L(1) = x^3 = 0 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 + 1 \cdot e_4$, so that the first column of the matrix should have entries $0, 0, 0, 1$. Similarly, $L(e_2) = 0 \cdot e_1 + (-1) \cdot e_2 + 0 \cdot e_3 + 0 \cdot e_4$, $L(e_3) = 0 \cdot e_1 + 0 \cdot e_2 + (-1) \cdot e_3 + 0 \cdot e_4$, and $L(e_4) = 1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 + 0 \cdot e_4$, so that the matrix is: $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

[3 marks].

We now compute $\det(\lambda I - A)$, using first $r_1 \leftrightarrow r_4$ (which negates the determinant) and then $r_4 \to r_4 + \lambda r_1$ (which leave the determinant unchanged), as follows:

$$\det \begin{pmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda+1 & 0 & 0 \\ 0 & 0 & \lambda+1 & 0 \\ -1 & 0 & 0 & \lambda \end{pmatrix} = -\det \begin{pmatrix} -1 & 0 & 0 & \lambda \\ 0 & \lambda+1 & 0 & 0 \\ 0 & 0 & \lambda+1 & 0 \\ \lambda & 0 & 0 & -1 \end{pmatrix}$$

$$= -\det \begin{pmatrix} -1 & 0 & 0 & \lambda \\ 0 & \lambda+1 & 0 & 0 \\ 0 & 0 & \lambda+1 & 0 \\ 0 & 0 & 0 & \lambda^2-1 \end{pmatrix} = -(-1)(\lambda+1)(\lambda+1)(\lambda^2-1),$$

which is $(\lambda - 1)(\lambda + 1)^3$. We therefore see that the possible eigenvalues are $\lambda = 1, -1$. When $\lambda = 1$, a vector $v = a + bx + cx^2 + dx^3$ is an eigenvector with eigenvalue 1 iff $L(v) = \lambda v$ iff $d - bx - cx^2 + ax^3 = a + bx + cx^2 + dx^3$ iff $d = a, -b = b, -c = c, a = d$ iff $a = d$ and $b = c = 0$ iff v is of the form $a + ax^3$ $(a \neq 0)$. When $\lambda = -1$, a vector $v = a + bx + cx^2 + dx^3$ is an eigenvector with eigenvalue -1 iff $L(v) = \lambda v$ iff $d - bx - cx^2 + ax^3 = -a - bx - cx^2 - dx^3$ iff $d = -a$ iff v is of the form $a + bx + cx^2 - ax^3$ $(a, b, c$ not all $0)$.

[6 marks]. *Whole question: seen similar (once) in exercises.*

9 marks in total for Question 6

SECTION B

7. A typical member of U is $\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$. Then $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in U$ by taking $a = b = d = 0$. If $\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right) \in U$ and $\lambda \in \mathbf{R}$ then $\lambda\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \lambda a & \lambda b \\ \lambda b & \lambda d \end{smallmatrix}\right) \in U$. Finally, if $\left(\begin{smallmatrix} a_1 & b_1 \\ b_1 & d_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ b_2 & d_2 \end{smallmatrix}\right) \in U$ then $\left(\begin{smallmatrix} a_1 & b_1 \\ b_1 & d_1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} a_2 & b_2 \\ b_2 & d_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1+a_2 & b_1+b_2 \\ b_1+b_2 & d_1+d_2 \end{smallmatrix}\right) \in U$. Hence, U is a subspace.

A typical member of W is $\left(\begin{smallmatrix} 0 & b \\ -b & 0 \end{smallmatrix}\right)$. Then $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in W$ by taking $b = 0$. If $\left(\begin{smallmatrix} 0 & b \\ -b & 0 \end{smallmatrix}\right) \in W$ and $\lambda \in \mathbf{R}$ then $\lambda\left(\begin{smallmatrix} 0 & b \\ -b & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \lambda b \\ \lambda(-b) & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \lambda b \\ -\lambda b & 0 \end{smallmatrix}\right) \in W$. Finally, if $\left(\begin{smallmatrix} 0 & b_1 \\ -b_1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & b_2 \\ -b_2 & 0 \end{smallmatrix}\right) \in W$ then $\left(\begin{smallmatrix} 0 & b_1 \\ -b_1 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & b_2 \\ -b_2 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & b_1+b_2 \\ -b_1-b_2 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & b_1+b_2 \\ -(b_1+b_2) & 0 \end{smallmatrix}\right) \in W$. Hence W is a subspace.

[4 marks].

Consider $M_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $M_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $M_3 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then any member of U, $\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$, can be written as a linear combination: $aM_1 + bM_2 + dM_3$, and so M_1, M_2, M_3 span U. Also, if $\lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3 = 0$, then $\left(\begin{smallmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and so $\lambda_1 = \lambda_2 = \lambda_3 = 0$, giving that M_1, M_2, M_3 are linearly independent. We conclude that $\{M_1, M_2, M_3\}$ is a basis for U and so U has dimension 3.

[Merely stating without justification that $\dim(U) = 3$ gets 1 mark; similarly for each of $\dim(W), \dim(U \cap W), \dim(U + W)$, below]

[3 marks].

A typical member of W looks like: $\left(\begin{smallmatrix} 0 & b \\ -b & 0 \end{smallmatrix}\right)$. Clearly, $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is a basis and so W has dimension 1.

[2 marks].

A matrix A is in $U \cap W$ if A is of the form $\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & b \\ -b & 0 \end{smallmatrix}\right)$ simultaneously, so that $a = d = 0$ and $b = -b$, giving $a = b = d = 0$, which is only possible when $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$; that is, $U \cap W$ consists only of $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and so has dimension 0.

[2 marks].

Any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $M_2(\mathbf{R})$ can be written as: $\left(\begin{smallmatrix} a & (b+c)/2 \\ (b+c)/2 & d \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & (b-c)/2 \\ (c-b)/2 & 0 \end{smallmatrix}\right)$, where the first addend is in U and the second in W. Therefore, anything in $M_2(\mathbf{R})$ can be written as an element of U plus an element of W, and so: $U + W = M_2(\mathbf{R})$, which has dimension 4 [since, for example, the standard basis: $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ has size 4]. [It is also acceptible to state and apply: $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$]

[3 marks].

Finally, it is true that $V = U \oplus W$, since we have both $V = U + W$ and $U \cap W = \{0\}$.

[1 mark].

15 marks in total for Question 7. *Unseen.*

8. The dual space V^* is defined to be the set of all linear maps from V to $\mathbf{R}$. Given $\theta, \phi \in V^*$, we can define $\theta + \phi$ by: $(\theta + \phi)(x) = \theta(x) + \phi(x)$, for all $x \in V$. Similarly, for $\lambda \in \mathbf{R}$, define $\lambda\theta$ by $(\lambda\theta)(x) = \lambda(\theta(x))$, for all $x \in V$. Given a basis $\{x_1, \ldots, x_n\}$ for V, the i-th member of the dual basis, ϕ_i, is defined to be the unique linear map from V to $\mathbf{R}$ such that $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$, for all $j \neq i$. Suppose $f \in V^*$; define $\lambda_j = f(x_j)$ for all j; then $(\lambda_1\phi_1 + \ldots + \lambda_n\phi_n)(x_j) = \lambda_j \cdot \phi_j(x_j)$ [since $\phi_i(x_j) = 0$, for all $j \neq i$] $= \lambda_j$ [since $\phi_j(x_j) = 1$]. Hence, f and $\lambda_1\phi_1 + \ldots + \lambda_n\phi_n$ both take the same values on each of $x_1, \ldots x_n$, giving that $f = \lambda_1\phi_1 + \ldots + \lambda_n\phi_n$ [since any linear map is completely determined by its values on a basis]. Hence, $\{\phi_1, \ldots \phi_n\}$ spans V^*. Now suppose that $\lambda_1\phi_1 + \ldots + \lambda_n\phi_n = 0$ for some $\lambda_1, \ldots, \lambda_n$. Then, for any j, $(\lambda_1\phi_1 + \ldots + \lambda_n\phi_n)(x_j) = 0$, and so $\lambda_j \cdot 1 = 0$; hence $\lambda_1 = \ldots = \lambda_n = 0$, and so $\phi_1, \ldots \phi_n$ are linearly independent. Hence $\{\phi_1, \ldots \phi_n\}$ is a basis for V^*.

[9 marks] *Bookwork*

In the given example, $\phi_1((x, y)) = ax + by \in V^*$ is defined to satisfy $\phi_1(v_1) = 1$ and $\phi_1(v_2) = 0$; similarly, $\phi_2((x, y)) = cx + dy \in V^*$ is defined to satisfy $\phi_2(v_1) = 0$ and $\phi_2(v_2) = 1$.

[2 marks]

That is: $2a - 5b = 1$ and $a - b = 0$, which has solution: $a = -1/3, b = -1/3$, so that ϕ_1 is defined by: $\phi_1((x, y)) = -(1/3)x - (1/3)y$. Similarly, $2c - 5d = 0$ and $c - d = 1$, which has solution: $c = 5/3, d = 2/3$, so that ϕ_2 is defined by: $\phi_2((x, y)) = (5/3)x + (2/3)y$. Hence, $\phi_1((2, 3)) = -5/3$ and $\phi_2((2, 3)) = 16/3$.

[4 marks] *Seen similar in exercises*

15 marks in total for Question 8

9. The law of inertia says that, if A is any real symmetric matrix, then there is an invertible matrix P such that $P^T AP$ is diagonal; further, all such diagonal representations of A have the same number p of positive entries, and n of negative entries. The *signature* is then $p - n$.

[4 marks]

For the given matrix A, we first construct $(A|I)$, and then diagonalise A by applying: step 1, $R_2 \to R_2 - R_1$ and $C_2 \to C_2 - C_1$; step 2, $R_3 \to R_3 - 3R_2$ and $C_3 \to C_3 - 3C_2$; only the column operation is applied to I:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & -3 & -5 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & -3 & 0 & 1 & 0 \\ 0 & -3 & -5 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & -1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right).$$

Taking

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix},$$

we then have: $D = P^T AP$, and so D is a diagonal form for A.

[8 marks].

The signature is $2 - 1 = 1$. A is not positive definite, since not all entries in the leading diagonal of D are positive.

[3 marks] *Seen similar in exercises.*

15 marks in total for Question 9

10.(i) The order k is the smallest integer ≥ 1 such that $g^k = e$. Two groups G and H are isomorphic if there is a map $\theta : G \to H$ which is bijective (1-1 and onto) and a homomorphism ($\theta(g_1 *_1 g_2) = \theta(g_1) *_2 \theta(g_2)$ for all $g_1, g_2 \in G$). Suppose there is a $g \in G$ of order k. Then k satisfies $g^k = e_G$ and is the smallest such. Then $\theta(g^k) = \theta(e_G) = e_H$, giving: $\theta(g)^k = e_H$, since θ is a homomorphism. So, $h^k = e_H$, where $h = \theta(g)$. Imagine that $h^r = e_H$ for some $1 \leq r < k$. Then $\theta^{-1}(h^r) = \theta^{-1}(e_H) = e_G$; that is: $g^r = \theta_G$, contradicting the assumption that k is the smallest integer ≥ 1 such that $g^k = e_G$. Hence order$(h) = k$, as required.

[7 marks] *Bookwork*

(ii) A presentation of the given group G is $\langle \sigma, \rho | \sigma^2 = \rho^n = e, \rho\sigma = \sigma\rho^{-1} \rangle$. Here, σ is a fixed reflection, and ρ is a rotation $2\pi/n$. A presentation of H is $\langle \rho | \rho^n = e \rangle$.

[5 marks] *From lectures*

Finally, R_6 contains an element (namely ρ) of order 6, whereas the orders of the elements in D_6 are: 1,2,2,2,3,3. So D_6 and R_6 are not isomorphic.

[3 marks] *Unseen*

15 marks in total for Question 10