SECTION A

1. Define what it means for a finite set of vectors to be a *basis* for a vector space V. Let V be the vector space of 2×2 matrices with entries in $\mathbf{R}$. Show that the set $\{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$ is a basis for V. Decide whether $\{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ is a basis for V. [8 marks]

2. Define the terms: *group, homomorphism, injective, surjective.*

Let G be the group of 2×2 matrices with real entries, nonzero determinant, and bottom left hand entry equal to zero, under the operation of matrix multiplication; let H be the group of positive real numbers under the operation of multiplication [you need not show that these are groups]. Let $\phi : G \to H$ be defined by

$$\phi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = a^2.$$

Show that ϕ is a homomorphism. State, giving reasons, whether ϕ is injective. State, giving reasons, whether ϕ is surjective. [9 marks]

3. Define the *rank* and *nullity* of a linear map $F : V \to W$. State the rank & nullity theorem.

Let $F : \mathbf{R}^4 \to \mathbf{R}^3$ be the linear mapping defined by:

$$F((x, y, s, t)) = (x + 3y + 2t, -x - 2y + s, 2x + 3y - 2t).$$

Find a basis for the image of F, and hence the rank of F. Find a basis for the kernel of F, and hence the nullity of F. Verify that the rank & nullity theorem holds in this case. [10 marks]

4. For any point A and angle α, let $\rho_{A,\alpha}$ denote rotation anticlockwise about A through angle α. For any line ℓ let σ_ℓ denote reflection in the line ℓ.

(i) Let ℓ and m be two lines which both pass through point A. Let α be the angle from ℓ to m. Show that $\sigma_m \sigma_\ell = \rho_{A,2\alpha}$.

(ii) Let s be any line through a point B, and let β be any angle. Use part (i) to find a line r through B such that $\rho_{B,\beta} = \sigma_s \sigma_r$ and a line t such that $\rho_{B,\beta} = \sigma_t \sigma_s$. Hence show that $\rho_{B,\beta}\sigma_s = \sigma_s \rho_{B,\beta} \iff \beta = 0, \pi$. [10 marks]

5. Let f be the bilinear form on $\mathbf{R}^2$ defined by

$$f((x_1, x_2), (y_1, y_2)) = x_1y_1 - 2x_1y_2 + x_2y_2.$$

Let $u_1 = (1, 2), u_2 = (-2, 3)$. Compute $f(u_1, u_1), f(u_1, u_2), f(u_2, u_1), f(u_2, u_2)$. Find the matrix A of f relative to the basis $\{u_1, u_2\}$. Find the matrix B of f relative to the basis $\{v_1, v_2\}$, where $v_1 = (-1, 5), v_2 = (4, 1)$.

Find the change of basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ and show that $B = P^T AP$. [9 marks]

6. Let V be the vector space of polynomials in x of degree at most 3 with coefficients in $\mathbf{R}$. Let the linear map $L : V \to V$ be defined by

$$L(a + bx + cx^2 + dx^3) = d - bx - cx^2 + ax^3.$$

Find M, the matrix representation of L with respect to the basis $\{1, x, x^2, x^3\}$. What are the eigenvalues and eigenvectors of ϕ? [9 marks]

SECTION B

7. Let V be the vector space of 2×2 matrices with entries in $\mathbf{R}$. Let

$$U = \{\begin{pmatrix} a & b \\ b & d \end{pmatrix} : a, b, d \in \mathbf{R}\},$$

and

$$W = \{\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} : b \in \mathbf{R}\}.$$

Show that U and W are subspaces of V. What are the dimensions of each of $U, W, U \cap W$ and $U + W$? Is it true or false that $V = U \oplus W$? [15 marks]

8. Suppose $\{x_1, \ldots, x_n\}$ is a basis for a vector space V. Describe the dual space V^* to V and describe how to define addition and scalar multiplication on V^* [you need not prove that V^* is a vector space]. Define the dual basis $\{\phi_1, \ldots, \phi_n\}$ to $\{x_1, \ldots, x_n\}$ and prove that it is a basis for V^*.

Consider the basis $\{v_1, v_2\}$ of $\mathbf{R}^2$, where $v_1 = (2, -5)$ and $v_2 = (1, -1)$. Find the dual basis $\{\phi_1, \phi_2\}$ to $\{v_1, v_2\}$. Compute $\phi_1((2, 3))$ and $\phi_2((2, 3))$. [15 marks]

9. State the law of inertia for a real symmetric matrix. Define the *signature* of a real symmetric matrix. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & -5 \end{pmatrix}.$$

Find a nonsingular matrix P such that $B = P^T AP$ is diagonal. Calculate the signature of A. Is A positive definite? [15 marks]

10. (i) Let g be an element of a group G. Define what is meant by the *order* of g. Define what it means for two groups G and H to be *isomorphic*. Prove that, if G and H are isomorphic and there is a $g \in G$ of order k, then there is an $h \in H$ of order k.

(ii) Let G be the dihedral group D_{2n} [the group of symmetries of a regular polygon of n sides]. Write down a presentation of G [you need not prove that it is a presentation]. Let H be R_n, the set of rotational symmetries of a regular polygon of n sides. Write down a presentation of H [you need not prove that it is a presentation]. Is D_6 isomorphic to R_6? [15 marks]