2MP43 (MATH243) September 1999 Examination

Time allowed: Two Hours and a Half

Candidates should attempt the whole of Section A and three questions from Section B

SECTION A

1. Let $f(x+iy) = u(x,y) + iv(x,y)$, where x, y, u and v are real. Write down the Cauchy-Riemann equations which hold where f is holomorphic.

Find real and imaginary parts of the function $f(z) = \bar{z}(z-1)$. Show that f satisfies both Cauchy-Riemann equations only at $z = 1$.

Find a holomorphic function on $\mathbf{C}$ with real part $u(x,y) = -xy$.

[10 marks]

2. Sketch the set $\{z : |z-2| = 1, \operatorname{Im} z > 0\} \subset \mathbf{C}$ and define a path γ that traverses this set once anticlockwise.

Evaluate $\int_\gamma (2 - \bar{z})\, dz$. [8 marks]

3. Evaluate the following integrals, giving brief reasons:

$$\int_{\gamma(i;2)} \frac{dz}{z^2-4}; \qquad \int_{\gamma(-1;2)} \frac{dz}{z^2-4}; \qquad \int_{\gamma(0;5)} \frac{dz}{z^2-4}.$$

Here $\gamma(a;r)$ denotes the circle, centre a and radius r, oriented anticlockwise.

[10 marks]

4. Find the 5-jet at 0 (the Taylor series up to and including the term z^5) of each of the following functions:

$$(i) \quad e^z \sin(z^2)\,; \qquad\qquad (ii) \quad \frac{\sinh z}{\cos z}\,.$$

[7 marks]

5. Determine the type of singularity exhibited by the function

$$f(z) = \frac{z^2(z^2 - 4\pi^2)}{(e^{iz} - 1)^2}$$

at (a) $z = 0$, (b) $z = -2\pi$, (c) $z = 4\pi$. If the singularity is removable determine the limiting value.

[10 marks]

6. (a) Find the residues of the function

$$f(z) = \frac{1}{z^2 - 10iz - 1}\,.$$

(b) Use the contour integration and the result of (a) to determine

$$\int_0^{2\pi} \frac{d\theta}{\sin\theta - 5}\,.$$

[10 marks]

SECTION B

7. (a) The function $\sin z$ is defined by $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$. Verify that

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y\,.$$

(b) Show that

$$\sin z = 0 \quad \Longleftrightarrow \quad z = \pi n\,, \;\; n \in \mathbf{Z}\,.$$

(c) Find all $z \in \mathbf{C}$ such that $\sin z = i$.

[15 marks]

8. (i) Find the radius of convergence R and the sum inside the circle of convergence of the series

$$\sum_{n=0}^{\infty} \frac{z^n}{3^n}\,.$$

Assuming the term-by-term differentiation is valid find $\sum_{n=1}^{\infty} n(\frac{z}{3})^{n-1}$.

(ii) Find the radius of convergence R of the series

$$\sum_{n=1}^{\infty} \frac{2^n z^n}{n^3}\,.$$

Determine the convergence or divergence of this series for $|z| = R$. [Make sure that your argument applies to all z with $|z| = R$.]

[15 marks]

9. (a) Sketch the annulus $\{z \in \mathbf{C} : 1 < |z-3| < 3\}$, and mark the poles of the function

$$f(z) = \frac{4}{z(z-4)}$$

on your sketch.

(b) Find the Laurent expansion of $f(z)$ valid in the above annulus.

(c) Determine whether this expansion converges at $z = 3 + i$.

[15 marks]

10. Sketch the path $\gamma_R : [0, \pi] \to \mathbf{C}$ defined by $\gamma_R(t) = Re^{it}$, where $R > 0$. Let a be a positive real number. Prove that

$$\int_{\gamma_R} \frac{z^2}{(z^2+2z+2)^2}\, dz \to 0 \quad \text{as} \quad R \to \infty .$$

By integrating $z^2/(z^2+2z+2)^2$ along a suitable contour, find

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+2x+2)^2}\, dx .$$

[15 marks]

11. Find the principal value of the integral

$$\int_{0}^{\infty} \frac{\cos x}{(x^2-4)(x^2+1)}\, dx$$

by integrating an appropriate holomorphic function round a large semicircle in the upper half-plane, indented both at 2 and -2.

State without proof any results you use on the limiting values of the integrals round the semicircular parts of the contour.

[15 marks]