SECTION A

1. Let $f(x+iy) = u(x,y) + iv(x,y)$, where x, y, u and v are real. Write down the Cauchy-Riemann equations which hold where f is holomorphic.

Find the real and imaginary parts of the function $f(z) = |z|^2 + i\overline{z}$. Show that f satisfies both Cauchy-Riemann equations if and only if $z = -i$.

Find a holomorphic function on $\mathbf{C}$ with the real part $u(x,y) = 2xy + 2x$.

[10 marks]

2. Sketch the path $\gamma : [0,2] \to \mathbf{C}$ given by

$$\gamma(t) = \begin{cases} (i-2)t\,, & 0 \le t \le 1 \\ i + 2(t-2)\,, & 1 \le t \le 2\,. \end{cases}$$

Evaluate $\int_\gamma \operatorname{Im}\overline{(i-z)}\, dz\,.$

[7 marks]

3. Evaluate the following integrals, giving brief reasons:

$$\int_{\gamma(3i;2)} \frac{dz}{z^2 - 2iz}\,; \qquad \int_{\gamma(-3i;2)} \frac{dz}{z^2 - 2iz}\,; \qquad \int_{\gamma(-2;4)} \frac{dz}{z^2 - 2iz}\,.$$

Here $\gamma(a;r)$ denotes the circle, centre a and radius r, oriented anticlockwise.

[10 marks]

4. Find the 5-jet at 0 (the Taylor series up to and including the term z^5) of each of the following functions:

$$(i) \quad e^{\cosh z^2 - 1}; \qquad\qquad (ii) \quad \frac{1}{2 - \sin^2(2z)}.$$

[8 marks]

5. Determine the type of singularity exhibited by the function

$$f(z) = \frac{\cos z \cdot \cot 2z}{(z-\pi)^4}$$

at (a) $z = 0$, (b) $z = \pi/2$, (c) $z = \pi$. If the singularity is a simple pole determine the residue, and if it is removable determine the limiting value.

[10 marks]

6. (a) Find the residues of the function

$$f(z) = \frac{1}{9z^2 + 82iz - 9}.$$

(b) Use contour integration and the result of (a) to determine

$$\int_0^{2\pi} \frac{dt}{9\sin t + 41}.$$

[10 marks]

SECTION B

7. (a) Write down the Laplace equation for a function $v(x,y)$ of two real variables.

(b) For what values of a real constant k can the function

$$v(x,y) = \cosh(kx) \cdot \sin(2y)$$

be the imaginary part of a function $f(z) = f(x+iy)$ holomorphic on $\mathbf{C}$?

(c) Write down the expression of the derivative of a holomorphic function in terms of the partial derivatives of its imaginary part. Apply this to determine the derivative $g(z) = f'(z)$ of a holomorphic function with the imaginary part v you obtained in (b). Express g in terms of z (not x and y).

(d) Show that, for the function g found in (c),

$$g(z) = 0 \qquad \Longleftrightarrow \qquad z = (\frac{\pi}{4} + \frac{\pi}{2}n)i \text{ for some integer } n\,.$$

[15 marks]

8. (i) Prove that, if $z^4 \neq \frac{1}{9}$, then

$$\sum_{n=0}^{r} 9^n z^{4n} = \frac{1 - 9^{r+1} z^{4r+4}}{1 - 9z^4}\,.$$

Hence show that $\sum\limits_{n=0}^{\infty} 9^n z^{4n}$ is convergent for $|z| < 1/\sqrt{3}$ and find a formula for the sum of this series.

Assuming that the term-by-term differentiation is valid, find the sum of the series

$$\sum_{n=1}^{\infty} n\, 9^n z^{4n}$$

for $|z| < 1/\sqrt{3}$.

(ii) Find the radius of convergence R of the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3 3^n} z^n\,.$$

Determine the convergence or divergence of this series for $|z| = R$. [Make sure that your argument applies to all z with $|z| = R$.]

[15 marks]

9. (a) Sketch the annulus $\{z \in \mathbf{C} : 4 < |z+5| < 6\}$, and mark the poles of the function

$$f(z) = \frac{2z}{z^2 - 1}$$

on your sketch.

(b) Find the Laurent expansion of $f(z)$ valid in the above annulus.

(c) Determine whether this expansion converges at $z = -5i$.

[15 marks]

10. Sketch the path $\gamma_R : [0, \pi] \to \mathbf{C}$ defined by $\gamma_R(t) = Re^{it}$, where $R > 0$. Prove that

$$\int_{\gamma_R} \frac{e^{5iz}}{(z^2 + 4z + 5)^2}\, dz \to 0 \quad \text{as} \quad R \to \infty\,.$$

By integrating $e^{5iz}/(z^2 + 4z + 5)^2$ along a suitable contour, find

$$\int_{-\infty}^{\infty} \frac{\cos(5x)}{(x^2 + 4x + 5)^2}\, dx\,.$$

[15 marks]

11. Let c be a positive constant. Find the principal value of the integral

$$\int_{0}^{\infty} \frac{x \sin 4x}{(x^2 + 1)(x^2 - c^2)}\, dx$$

by integrating an appropriate holomorphic function round a large semicircle in the upper half-plane, indented both at c and $-c$.

State without proof any results you use on the limiting values of the integrals round the semicircular parts of the contour.

[15 marks]