MATH243 September 2002 Examination

Time allowed: Two Hours and a Half

Canditates should attempt the whole of Section A and three questions from Section B

SECTION A

1. Let $f(x+iy) = u(x,y) + iv(x,y)$, where x, y, u and v are real. Write down the Cauchy-Riemann equations which hold where f is holomorphic.

Find the real and imaginary parts of the function $f(z) = z(\bar{z} - 4i)$. Show that f satisfies both Cauchy-Riemann equations only at $z = 0$.

Find a holomorphic function on $\mathbf{C}$ with the real part $v(x,y) = 3(x^2 - y^2)$.

[10 marks]

2. Sketch the path $\gamma : [-1,1] \to \mathbf{C}$ given by

$$\gamma(t) = \begin{cases} 2it\,, & -1 \le t \le 0 \\ -3t\,, & 0 \le t \le 1\,. \end{cases}$$

Evaluate $\int_\gamma (\operatorname{Im} z)^3\, dz\,.$

[7 marks]

3. Evaluate the following integrals, giving brief reasons:

$$\int_{\gamma(0;1)} \frac{dz}{4-z^2}\,; \qquad \int_{\gamma(2;3)} \frac{dz}{4-z^2}\,; \qquad \int_{\gamma(-i;5)} \frac{dz}{4-z^2}\,.$$

Here $\gamma(a;r)$ denotes the circle, centre a and radius r, oriented anticlockwise.

[10 marks]

4. Find the 5-jet at 0 (the Taylor series up to and including the term z^5) of each of the following functions:

$$(i) \quad \cosh(z+z^4)\,; \qquad\qquad (ii) \quad \frac{z^3+2}{1+\sin(z^2)}\,.$$

[8 marks]

5. Determine the type of singularity exhibited by the function

$$f(z) = \frac{z\cot z}{(2z-\pi)^3}$$

at (a) $z=\pi$, (b) $z=0$, (c) $z=\pi/2$. If the singularity is a simple pole determine the residue, and if it is removable determine the limiting value.

[10 marks]

6. (a) Find the residues of the function

$$f(z) = \frac{1}{7z^2 + 50z + 7} .$$

(b) Use contour integration and the result of (a) to determine

$$\int_0^{2\pi} \frac{d\theta}{25 + 7\cos\theta} .$$

[10 marks]

SECTION B

7. (a) Write down the Laplace equation for a function $u(x, y)$ of two real variables.

(b) For what value of the constant k can the function

$$u(x, y) = kx^2 + 3y^2$$

be the real part of a function $f(z) = f(x + iy)$ holomorphic on $\mathbf{C}$?

(c) Write down the expression of the derivative of a holomorphic function in terms of the partial derivatives of its real part. Apply this to determine the derivative $g(z) = f'(z)$ of a holomorphic function with the real part u you obtained in (b). Express g in terms of z (not x and y).

(d) Show that, for the function g found in (c),

$$e^{ig(z)} = -i \qquad \Longleftrightarrow \qquad z = \frac{\pi}{12} + \frac{\pi}{3}n \text{ for some integer } n .$$

[15 marks]

8. (i) Find the radius of convergence R and the sum inside the circle of convergence of the series

$$\sum_{n=0}^{\infty} \frac{z^n}{2^{2n}}.$$

Assuming the term-by-term differentiation is valid find $\sum_{n=1}^{\infty} n(z/4)^{n-1}$.

(ii) Find the radius of convergence R of the series

$$\sum_{n=1}^{\infty} \frac{(-5)^n z^n}{3n^4}.$$

Determine the convergence or divergence of this series for $|z| = R$. [Make sure that your argument applies to all z with $|z| = R$.]

[15 marks]

9. (a) Sketch the annulus $\{z \in \mathbf{C} : 2 < |z-3| < 7\}$, and mark the poles of the function

$$f(z) = \frac{9}{z^2 - z - 20}$$

on your sketch.

(b) Find the Laurent expansion of $f(z)$ valid in the above annulus.

(c) Determine whether this expansion converges at $z = 10$.

[15 marks]

10. Sketch the path $\gamma_R : [0,\pi] \to \mathbf{C}$ defined by $\gamma_R(t) = Re^{it}$, where $R > 0$. Prove that

$$\int_{\gamma_R} \frac{z^2}{(9z^2+4)^2}\, dz \to 0 \quad \text{as} \quad R \to \infty\,.$$

By integrating $6z^2/(z^2+25)^2$ along a suitable contour, find

$$\int_{-\infty}^{\infty} \frac{6x^2 - 11x}{(x^2+25)^2}\, dx\,.$$

[15 marks]

11. Find the principal value of the integral

$$\int_{0}^{\infty} \frac{\cos 5x}{(x^2-1)(x^2+9)}\, dx$$

by integrating an appropriate holomorphic function round a large semicircle in the upper half-plane, indented both at 1 and -1.

State without proof any results you use on the limiting values of the integrals round the semicircular parts of the contour.

[15 marks]