SECTION A

1. Let $f(x+iy) = u(x,y) + iv(x,y)$, where x, y, u and v are real. Write down the Cauchy-Riemann equations which hold where f is holomorphic.

Find the real and imaginary parts of the function $f(z) = \bar{z}(z+i)$. Show that f satisfies both Cauchy-Riemann equations only at $z = -i$.

Find a holomorphic function on $\mathbf{C}$ with the imaginary part $v(x,y) = 2xy - x$.

[10 marks]

2. Sketch the path $\gamma : [-1,2] \to \mathbf{C}$ given by

$$\gamma(t) = \begin{cases} -it\,, & -1 \le t \le 0 \\ (2i-1)t\,, & 0 \le t \le 2\,. \end{cases}$$

Evaluate $\int_\gamma (\operatorname{Re} z)^4 \, dz$.

[7 marks]

3. Evaluate the following integrals, giving brief reasons:

$$\int_{\gamma(i;1)} \frac{dz}{4z^2-1}\,; \qquad \int_{\gamma(1;1)} \frac{dz}{4z^2-1}\,; \qquad \int_{\gamma(0;3)} \frac{dz}{4z^2-1}\,.$$

Here $\gamma(a;r)$ denotes the circle, centre a and radius r, oriented anticlockwise.

[10 marks]

4. Find the 5-jet at 0 (the Taylor series up to and including the term z^5) of each of the following functions:

$$(i) \quad (z^2-1)e^{z^2}; \qquad\qquad (ii) \quad \frac{\sin(2z)}{\cosh z}.$$

[8 marks]

5. Determine the type of singularity exhibited by the function

$$f(z) = \frac{(2z-\pi)(2z+\pi)^2}{\cos^2 z}$$

at (a) $z = 3\pi/2$, (b) $z = \pi/2$, (c) $z = -\pi/2$. If the singularity is a simple pole determine the residue, and if it is removable determine the limiting value.

[10 marks]

6. (a) Find the residues of the function

$$f(z) = \frac{1}{2z^2 - 5iz - 2}.$$

(b) Use contour integration and the result of (a) to determine

$$\int_0^{2\pi} \frac{d\theta}{4\sin\theta - 5}.$$

[10 marks]

SECTION B

7. (a) Write down the Laplace equation for a function $v(x, y)$ of two real variables.

(b) For what value of the constant k can the function

$$v(x, y) = -4x^2 + ky^2$$

be the imaginary part of a function $f(z) = f(x + iy)$ holomorphic on $\mathbf{C}$?

(c) Write down the expression of the derivative of a holomorphic function in terms of the partial derivatives of its imaginary part. Apply this to determine the derivative $g(z) = f'(z)$ of a holomorphic function with the imaginary part v you obtained in (b). Express g in terms of z (not x and y).

(d) Show that, for the function g found in (c),

$$e^{g(z)} = -1 \qquad \Longleftrightarrow \qquad z = \frac{\pi}{8} + \frac{\pi}{4}n \text{ for some integer } n\,.$$

[15 marks]

8. (i) Find the radius of convergence R and the sum inside the circle of convergence of the series

$$\sum_{n=0}^{\infty} \frac{z^n}{6^n}.$$

Assuming the term-by-term differentiation is valid find $\sum_{n=1}^{\infty} n(z/6)^{n-1}$.

(ii) Find the radius of convergence R of the series

$$\sum_{n=1}^{\infty} \frac{(-3)^n z^n}{n^5}.$$

Determine the convergence or divergence of this series for $|z| = R$. [Make sure that your argument applies to all z with $|z| = R$.]

[15 marks]

9. (a) Sketch the annulus $\{z \in \mathbf{C} : 1 < |z-4| < 3\}$, and mark the poles of the function

$$f(z) = \frac{2}{z^2 - 4z + 3}$$

on your sketch.

(b) Find the Laurent expansion of $f(z)$ valid in the above annulus.

(c) Determine whether this expansion converges at $z = 7$.

[15 marks]

10. Sketch the path $\gamma_R : [0, \pi] \to \mathbf{C}$ defined by $\gamma_R(t) = Re^{it}$, where $R > 0$. Prove that

$$\int_{\gamma_R} \frac{z^2}{(z^2+4)^2}\, dz \to 0 \quad \text{as} \quad R \to \infty\,.$$

By integrating $3z^2/(z^2+4)^2$ along a suitable contour, find

$$\int_{-\infty}^{\infty} \frac{3x^2 - 5x}{(x^2+4)^2}\, dx\,.$$

[15 marks]

11. Find the principal value of the integral

$$\int_{0}^{\infty} \frac{x \sin 4x}{(x^2-4)(x^2+1)}\, dx$$

by integrating an appropriate holomorphic function round a large semicircle in the upper half-plane, indented both at 2 and -2.

State without proof any results you use on the limiting values of the integrals round the semicircular parts of the contour.

[15 marks]