SECTION A

1. Let $f(x+iy) = u(x,y) + iv(x,y)$, where x, y, u and v are real. Write down the Cauchy-Riemann equations which hold where f is holomorphic.

Find the real and imaginary parts of the function $f(z) = (z - \bar{z})^3$. Show that f satisfies both Cauchy-Riemann equations if and only if z is real.

Find a holomorphic function on $\mathbf{C}$ with the imaginary part $v(x,y) = (y^2 - x^2)/2$.

[10 marks]

2. Sketch the path $\gamma : [-2,1] \to \mathbf{C}$ given by

$$\gamma(t) = \begin{cases} 2i - t\,, & -2 \le t \le 0 \\ i(t+2)\,, & 0 \le t \le 1\,. \end{cases}$$

Evaluate $\int\limits_{\gamma} (\mathrm{Re}\,(i\bar{z})\, dz\,.$

[8 marks]

3. Evaluate the following integrals, giving brief reasons:

$$\int\limits_{\gamma(2;1)} \frac{dz}{4z^2+9}\,; \qquad \int\limits_{\gamma(-2i;1)} \frac{dz}{4z^2+9}\,; \qquad \int\limits_{\gamma(-1;e)} \frac{dz}{4z^2+9}\,.$$

Here $\gamma(a;r)$ denotes the circle, centre a and radius r, oriented anticlockwise.

[10 marks]

4. Find the 5-jet at 0 (the Taylor series up to and including the term z^5) of each of the following functions:

$$(i) \quad \cosh(3z) \cdot \sin z\,; \qquad\qquad (ii) \quad \frac{z^2 - 1}{2 - e^{z^2}}\,.$$

[7 marks]

5. Determine the type of singularity exhibited by the function

$$f(z) = \frac{(\pi - 2z)(\sin z + 1)}{\cos^2 z}$$

at (a) $z = -3\pi/2$, (b) $z = -\pi/2$, (c) $z = \pi/2$. If the singularity is a simple pole determine the residue, and if it is removable determine the limiting value.

[10 marks]

6. (a) Find the residues of the function

$$f(z) = \frac{1}{5z^2 - 26z + 5}\,.$$

(b) Use contour integration and the result of (a) to determine

$$\int_0^{2\pi} \frac{d\theta}{5\cos\theta - 13}\,.$$

[10 marks]

SECTION B

7. (a) Write down the Laplace equation for a function $u(x, y)$ of two real variables.

(b) For what value of the positive constant k can the function

$$u(x, y) = \frac{1}{5} e^{ky} \cos 5x$$

be the real part of a function $f(z) = f(x + iy)$ holomorphic on $\mathbf{C}$?

(c) Write down the expression of the derivative of a holomorphic function in terms of the partial derivatives of its real part. Apply this to determine the derivative $g(z) = f'(z)$ of a holomorphic function with the real part u you obtained in (b). Express g in terms of z (not x and y).

(d) Show that, for the function g found in (c),

$$g(z) = 1 \qquad \Longleftrightarrow \qquad z = -\frac{\pi}{10} + \frac{2\pi}{5} n \text{ for some integer } n\,.$$

[15 marks]

8. (i) Prove that, if $z \neq 0, 3$,

$$\sum_{n=0}^{r} (3/z)^n = \frac{1 - (3/z)^{r+1}}{1 - \frac{3}{z}}.$$

Hence show that $\sum_{n=0}^{\infty} (3/z)^n$ is convergent for $|z| > 3$ and find a formula for the sum of this series.

Assuming that the term-by-term differentiation is valid, find the sum of the series

$$\sum_{n=1}^{\infty} n(3/z)^{n+1}$$

for $|z| > 3$.

(ii) Find the radius of convergence R of the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^5 + 3n} z^n.$$

Determine the convergence or divergence of this series for $|z| = R$. [Make sure that your argument applies to all z with $|z| = R$.]

[15 marks]

9. (a) Sketch the annulus $\{z \in \mathbf{C} : 2 < |z - 1| < 3\}$, and mark the poles of the function

$$f(z) = \frac{1}{z^2 - 7z + 12}$$

on your sketch.

(b) Find the Laurent expansion of $f(z)$ valid in the above annulus.

(c) Determine whether this expansion converges at $z = -2$.

[15 marks]

10. Sketch the path $\gamma_R : [0, \pi] \to \mathbf{C}$ defined by $\gamma_R(t) = Re^{it}$, where $R > 0$. Prove that

$$\int_{\gamma_R} \frac{ze^{2iz}}{(4z^2+1)^2}\, dz \to 0 \quad \text{as} \quad R \to \infty\,.$$

By integrating $ze^{2iz}/(4z^2+1)^2$ along a suitable contour, find

$$\int_0^\infty \frac{x \sin 2x}{(4x^2+1)^2}\, dx\,.$$

[15 marks]

11. Find the principal value of the integral

$$\int_0^\infty \frac{\cos 3x}{16 - x^4}\, dx$$

by integrating an appropriate holomorphic function round a large semicircle in the upper half-plane, indented both at 2 and -2.

State without proof any results you use on the limiting values of the integrals round the semicircular parts of the contour.

[15 marks]