2MP43 (MATH243) January 1999 Examination

Time allowed: Two Hours and a Half

Candidates should attempt the whole of Section A and three questions from Section B

SECTION A

1. Let $f(x+iy) = u(x,y) + iv(x,y)$, where x, y, u and v are real. Write down the Cauchy-Riemann equations which hold where f is holomorphic.

Find real and imaginary parts of the function $f(z) = z(\bar{z}+1)$. Show that f satisfies both Cauchy-Riemann equations only at $z = 0$.

Find a holomorphic function on $\mathbf{C}$ with imaginary part $v(x,y) = -xy$.

[10 marks]

2. Sketch the path $\gamma : [-1,2] \to \mathbf{C}$ given by

$$\gamma(t) = \begin{cases} (1-i)t\,, & -1 \le t \le 0 \\ t\,, & 0 \le t \le 2\,. \end{cases}$$

Evaluate $\int_\gamma (\operatorname{Im} z)^2 \, dz$.

[8 marks]

3. Evaluate the following integrals, giving brief reasons:

$$\int_{\gamma(0;1)} \frac{dz}{z^2+4}\,; \qquad \int_{\gamma(-i;2)} \frac{dz}{z^2+4}\,; \qquad \int_{\gamma(0;4)} \frac{dz}{z^2+4}\,.$$

Here $\gamma(a;r)$ denotes the circle, centre a and radius r, oriented anticlockwise.

[10 marks]

4. Find the 5-jet at 0 (the Taylor series up to and including the term z^5) of each of the following functions:

$$(i) \quad e^{z^3} \sin z\,; \qquad\qquad (ii) \quad \frac{2-z^2}{\cos z}\,.$$

[7 marks]

5. Determine the type of singularity exhibited by the function

$$f(z) = (2 + \frac{\pi}{z}) \tan^2 z$$

at (a) $z = -\pi/2$, (b) $z = 0$, (c) $z = \pi/2$. If the singularity is a simple pole determine the residue, and if it is removable determine the limiting value.

[10 marks]

6. (a) Find the residues of the function

$$f(z) = \frac{1}{z^2 + 4z + 1}\,.$$

(b) Use contour integration and the result of (a) to determine

$$\int_0^{2\pi} \frac{d\theta}{\cos\theta + 2}\,.$$

[10 marks]

SECTION B

7. (a) Write down the Laplace equation for a function $u(x, y)$ of two real variables.

(b) For what value(s) of the constant k can the function

$$u(x, y) = e^{-4y} \cos kx$$

be the real part of a function $f(z) = f(x + iy)$ holomorphic on $\mathbf{C}$?

(c) Write down the expression of the derivative of a holomorphic function in terms of the partial derivatives of its real part. Apply this to determine the derivative $g(z) = f'(z)$ of a holomorphic function with the real part you found in (b). Express g in terms of z (not x and y).

(d) For the function g found in (c), solve the equation

$$g(z) = -4i\,.$$

[All the solutions must be found.]

[15 marks]

8. (i) Prove that, if $z \neq 0, -1$,

$$\sum_{n=0}^{r} (-1)^n z^{-n} = \frac{1 + (-1)^r z^{-r-1}}{1 + z^{-1}}.$$

Hence show that $\sum_{n=0}^{\infty} (-1)^n z^{-n}$ is convergent for $|z| > 1$ and find a formula for $\sum_{k=0}^{-\infty} (-1)^k z^k$.

Describe the behaviour of the last series for $|z| \leq 1$.

(ii) Find the radius of convergence R of the series

$$\sum_{n=1}^{\infty} \frac{2^n - 1}{n^4} z^n .$$

Determine the convergence or divergence of this series for $|z| = R$. [Make sure that your argument applies to all z with $|z| = R$.]

[15 marks]

9. (a) Sketch the annulus $\{z \in \mathbf{C} : 2 < |z - 1| < 4\}$, and mark the poles of the function

$$f(z) = \frac{6}{z^2 - 9}$$

on your sketch.

(b) Find the Laurent expansion of $f(z)$ valid in the above annulus.

(c) Determine whether this expansion converges at $z = -1$.

[15 marks]

10. Sketch the path $\gamma_R : [0, \pi] \to \mathbf{C}$ defined by $\gamma_R(t) = Re^{it}$, where $R > 0$. Let a be a positive real number. Prove that

$$\int_{\gamma_R} \frac{ze^{iz}}{(z^2 + a^2)^2}\, dz \to 0 \quad \text{as} \quad R \to \infty\,.$$

By integrating $(ze^{iz})/(z^2 + a^2)^2$ along a suitable contour, find

$$\int_0^\infty \frac{x \sin x}{(x^2 + a^2)^2}\, dx\,.$$

[15 marks]

11. Find the principal value of the integral

$$\int_0^\infty \frac{\cos x}{x^4 - 1}\, dx$$

by integrating an appropriate holomorphic function round a large semicircle in the upper half-plane, indented both at 1 and -1.

State without proof any results you use on the limiting values of the integrals round the semicircular parts of the contour.

[15 marks]