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SECTION A

1. Let $f(x+iy) = u(x,y)+iv(x,y)$, where x, y, u and v are real. Write down the Cauchy-Riemann equations which hold where f is complex differentiable.

Find the real and imaginary parts of the function $f(z) = i|z|^2 - z^2$. Show that f satisfies both Cauchy-Riemann equations if and only if $z = 0$.

Find a holomorphic function on $\mathbf{C}$ with the real part $u(x,y) = y - 2xy$.

[10 marks]

2. Sketch the path $\gamma : [-1,1] \to \mathbf{C}$ given by

$$\gamma(t) = \begin{cases} 2t\,, & -1 \le t \le 0\,, \\ -t - it\,, & 0 \le t \le 1\,. \end{cases}$$

Evaluate $\int_\gamma \operatorname{Im} z \, dz$.

[7 marks]

3. Evaluate the following integrals, in each case giving a brief justification for your answer:

$$\int_{\gamma(i;1)} \frac{dz}{z^2-4}\,; \qquad \int_{\gamma(-6;5)} \frac{dz}{z^2-4}\,; \qquad \int_{\gamma(-i;10)} \frac{dz}{z^2-4}\,.$$

Here $\gamma(a;r)$ denotes the circle, centre a and radius r, oriented anticlockwise.

[10 marks]

4. Find the Taylor series at the origin up to and including the term z^5 (5-jet) of each of the following functions:

$$(i) \quad \sinh(2z^4 + z)\,; \qquad\qquad (ii) \quad \frac{1}{\cos^2(z)}\,.$$

[8 marks]

5. Determine the type of singularity exhibited by the function

$$f(z) = \frac{\sin 2z}{(2z+\pi)z^3\cos z}$$

at (a) $z = -\pi/2$, (b) $z = 0$, (c) $z = \pi/2$. If the singularity is a simple pole determine the residue, and if it is removable determine the limiting value.

[10 marks]

6. (a) Find the residues of the function

$$f(z) = \frac{1}{z^2 - 5z + 1}\,.$$

(b) Use contour integration and the result of (a) to determine

$$\int_0^{2\pi} \frac{d\theta}{5 - 2\cos\theta}\,.$$

[10 marks]

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SECTION B

7. (a) Write down the Laplace equation for a function $v(x, y)$ of two real variables.

(b) For what value of a real constant $k > 0$ can the function

$$v(x, y) = e^{2x} \cdot \sin(ky)$$

be the imaginary part of a function $f(z) = f(x + iy)$ holomorphic on $\mathbf{C}$?

(c) Write down the expression of the derivative of a holomorphic function in terms of the partial derivatives of its imaginary part only. Apply this to determine the derivative $g(z) = f'(z)$ of a holomorphic function with the imaginary part v you obtained in (b). Express g in terms of z only (not x and y).

(d) Show that, for the function g found in (c),

$$g(z) = -2 \qquad \Longleftrightarrow \qquad z = \frac{i\pi}{2}(2n + 1) \text{ for some integer } n\,.$$

[15 marks]

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8. (i) Prove that, if $z^2 \neq -i$, then

$$\sum_{n=0}^{m} \left(iz^2\right)^n = \frac{iz^2(i^m z^{2m}) - 1}{iz^2 - 1}.$$

Hence show that $\sum_{n=0}^{\infty} i^n z^{2n}$ is convergent for $|z| < 1$ and find a formula for the sum of this series.

Assuming that the term-by-term differentiation is valid, find the sum of the series

$$\sum_{n=1}^{\infty} n i^n z^{2n}$$

for $|z| < 1$.

(ii) Find the radius of convergence R of the series

$$\sum_{n=0}^{\infty} \frac{(2i)^n}{n^2 + 1} z^n .$$

Determine the convergence or divergence of this series for $|z| = R$. Make sure that your argument applies to all z with $|z| = R$.

[15 marks]

9. (a) Sketch the annulus $\{z \in \mathbf{C} : 2 < |z - 2i| < 3\}$, and mark the poles of the function

$$f(z) = \frac{10}{z^2 - 5iz}$$

on your sketch.

(b) Find the Laurent expansion of $f(z)$ valid in the above annulus.

(c) Determine whether this expansion converges at $z = -2 + i$.

[15 marks]

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10. Sketch the path $\gamma_R : [-\pi; 0] \to \mathbf{C}$ defined by $\gamma_R(t) = Re^{-it}$, where $R > 0$. Prove that

$$\int_{\gamma_R} \frac{e^{iz}}{(z^2+1)^2}\, dz \to 0 \quad \text{as} \quad R \to \infty\,.$$

By integrating $\dfrac{e^{iz}}{(z^2+1)^2}$ along a suitable contour, find $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\, dx\,.$

[15 marks]

11. Find the principal value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{(x+1)(x^2+2x+2)}\, dx$$

by integrating an appropriate holomorphic function round a large semicircle in the upper half-plane, indented at $z = -1$.

State without proof any results you use on the limiting values of the integrals round the semicircular parts of the contour.

[15 marks]