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SECTION A

1. Let $f(x + iy) = u(x, y) + iv(x, y)$, where x, y, u and v are real. Write down the Cauchy-Riemann equations which hold where f is holomorphic.

Find the real and imaginary parts of the function $f(z) = |z|^2 - z^2$. Show that f satisfies both Cauchy-Riemann equations if and only if $z = 0$.

Find a holomorphic function on \mathbf{C} with the imaginary part $v(x, y) = 3x^2 - 3y^2$.

[10 marks]

2. Sketch the path $\gamma : [-1, 2] \rightarrow \mathbf{C}$ given by

$$\gamma(t) = \begin{cases} it, & -1 \leq t \leq 1, \\ 2t - 2 + i, & 1 \leq t \leq 2. \end{cases}$$

Evaluate $\int_{\gamma} \operatorname{Im} z \, dz$.

[7 marks]

3. Evaluate the following integrals, in each case giving a brief justification for your answer:

$$\int_{\gamma(6;1)} \frac{dz}{z^2 + 36}; \quad \int_{\gamma(-5i;2)} \frac{dz}{z^2 + 36}; \quad \int_{\gamma(0;10)} \frac{dz}{z^2 + 36}.$$

Here $\gamma(a; r)$ denotes the circle, centre a and radius r , oriented anticlockwise.

[10 marks]



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4. Find the 5-jet at 0 (the Taylor series up to and including the term z^5) of each of the following functions:

(i) $\sin(z^4 - z)$; (ii) $\frac{1}{\cosh^2(2z)}$.

[8 marks]

5. Determine the type of singularity exhibited by the function

$$f(z) = \frac{\cos z}{(2z - \pi)^2 \cdot \sin 2z}$$

at (a) $z = 0$, (b) $z = \pi/2$, (c) $z = -\pi/2$. If the singularity is a simple pole determine the residue, and if it is removable determine the limiting value.

[10 marks]

6. (a) Find the residues of the function

$$f(z) = \frac{1}{z^2 + 3z + 1}.$$

(b) Use contour integration and the result of (a) to determine

$$\int_0^{2\pi} \frac{d\theta}{2 \cos \theta + 3}.$$

[10 marks]



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SECTION B

7. (a) Write down the Laplace equation for a function $u(x, y)$ of two real variables.

(b) For what values of a real constant $k > 0$ can the function

$$u(x, y) = e^{kx} \cdot \sin(2y)$$

be the real part of a function $f(z) = f(x + iy)$ holomorphic on \mathbf{C} ?

(c) Write down the expression of the derivative of a holomorphic function in terms of the partial derivatives of its real part. Apply this to determine the derivative $g(z) = f'(z)$ of a holomorphic function with the real part u you obtained in (b). Express g in terms of z (not x and y).

(d) Show that, for the function g found in (c),

$$g(z) = 2 \quad \iff \quad z = i\left(\frac{\pi}{4} + \pi n\right) \text{ for some integer } n.$$

[15 marks]



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8. (i) Prove that, if $z^4 \neq \frac{1}{16}$, then

$$\sum_{n=0}^r (16z^4)^n = \frac{1 - 16^{r+1}z^{4r+4}}{1 - 16z^4}.$$

Hence show that $\sum_{n=0}^{\infty} 16^n z^{4n}$ is convergent for $|z| < \frac{1}{2}$ and find a formula for the sum of this series.

Assuming that the term-by-term differentiation is valid, find the sum of the series

$$\sum_{n=1}^{\infty} n 16^n z^{4n}$$

for $|z| < \frac{1}{2}$.

(ii) Find the radius of convergence R of the series

$$\sum_{n=0}^{\infty} \frac{5^n}{n^2 + 3} z^n.$$

Determine the convergence or divergence of this series for $|z| = R$. [Make sure that your argument applies to all z with $|z| = R$.]

[15 marks]

9. (a) Sketch the annulus $\{z \in \mathbf{C} : 1 < |z - 4| < 4\}$, and mark the poles of the function

$$f(z) = \frac{10}{z^2 - 5z}$$

on your sketch.

- (b) Find the Laurent expansion of $f(z)$ valid in the above annulus.
(c) Determine whether this expansion converges at $z = 1 - 3i$.

[15 marks]



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10. Sketch the path $\gamma_R : [0, \pi] \rightarrow \mathbf{C}$ defined by $\gamma_R(t) = Re^{it}$, where $R > 0$.

Prove that

$$\int_{\gamma_R} \frac{e^{4iz}}{(z^2 + 8z + 17)^2} dz \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

By integrating $e^{4iz}/(z^2 + 8z + 17)^2$ along a suitable contour, find

$$\int_{-\infty}^{\infty} \frac{\cos(4x)}{(x^2 + 8x + 17)^2} dx.$$

[15 marks]

11. Let c be a positive constant. Find the principal value of the integral

$$\int_0^{\infty} \frac{x \sin(2x)}{(x^2 - c^2)(x^2 + 16)} dx$$

by integrating an appropriate holomorphic function round a large semicircle in the upper half-plane, indented both at $z = c$ and $z = -c$.

State without proof any results you use on the limiting values of the integrals round the semicircular parts of the contour.

[15 marks]