MATH243 January 2002 Examination

Time allowed: Two Hours and a Half

Candidates should attempt the whole of Section A
and three questions from Section B

SECTION A

1. Let $f(x+iy) = u(x,y) + iv(x,y)$, where x, y, u and v are real. Write down the Cauchy-Riemann equations which hold where f is holomorphic.

Find the real and imaginary parts of the function $f(z) = \frac{1}{4}(z+\bar{z})^4$. Show that f satisfies both Cauchy-Riemann equations if and only if z is purely imaginary.

Find a holomorphic function on $\mathbf{C}$ with the real part $u(x,y) = 3xy$.

[10 marks]

2. Sketch the path $\gamma : [-2,1] \to \mathbf{C}$ given by

$$\gamma(t) = \begin{cases} 2+it\,, & -1 \le t \le 1 \\ i+2t\,, & 1 \le t \le 2\,. \end{cases}$$

Evaluate $\int_\gamma \operatorname{Im}(\bar{iz})\,dz\,.$

[8 marks]

3. Evaluate the following integrals, giving brief reasons:

$$\int_{\gamma(i;1)} \frac{dz}{9z^2+4}\,; \qquad \int_{\gamma(0;\pi)} \frac{dz}{9z^2+4}\,; \qquad \int_{\gamma(e;2)} \frac{dz}{9z^2+4}\,.$$

Here $\gamma(a;r)$ denotes the circle, centre a and radius r, oriented anticlockwise.

[10 marks]

4. Find the 5-jet at 0 (the Taylor series up to and including the term z^5) of each of the following functions:

$$(i) \quad \cos(\sin z)\,; \qquad\qquad (ii) \quad \frac{e^{z^2}}{\cosh(2z)}\,.$$

[7 marks]

5. Determine the type of singularity exhibited by the function

$$f(z) = \frac{(2z+\pi)\tan z}{z^3}$$

at (a) $z=0$, (b) $z=-\pi/2$, (c) $z=\pi/2$. If the singularity is a simple pole determine the residue, and if it is removable determine the limiting value.

[10 marks]

6. (a) Find the residues of the function

$$f(z) = \frac{1}{12z^2 + 25iz - 12}\,.$$

(b) Use contour integration and the result of (a) to determine

$$\int_0^{2\pi} \frac{d\theta}{25 + 24\sin\theta}\,.$$

[10 marks]

SECTION B

7. (a) Write down the Laplace equation for a function $u(x, y)$ of two real variables.

(b) For what value of the positive constant k can the function

$$v(x, y) = e^{ky}\cos x + x$$

be the imaginary part of a function $f(z) = f(x + iy)$ holomorphic on $\mathbf{C}$?

(c) Write down the expression of the derivative of a holomorphic function in terms of the partial derivatives of its imaginary part. Apply this to determine the derivative $g(z) = f'(z)$ of a holomorphic function with the imaginary part v you obtained in (b). Express g in terms of z (not x and y).

(d) Show that, for the function g found in (c),

$$g(z) = 0 \qquad \Longleftrightarrow \qquad z = \frac{\pi}{2} + 2\pi n \text{ for some integer } n\,.$$

[15 marks]

8. (i) Prove that, if $z \neq 0, \pm 1$,

$$\sum_{n=0}^{r} (1/z^2)^n = \frac{1-(1/z^2)^{r+1}}{1-\frac{1}{z^2}}.$$

Hence show that $\sum_{n=0}^{\infty} z^{-2n}$ is convergent for $|z| > 1$ and find a formula for the sum of this series.

Assuming that the term-by-term differentiation is valid, find the sum of the series

$$\sum_{n=1}^{\infty} nz^{-2n-1}$$

for $|z| > 1$.

(ii) Find the radius of convergence R of the series

$$\sum_{n=1}^{\infty} \frac{3^n}{2n^4+4n-1} z^n .$$

Determine the convergence or divergence of this series for $|z| = R$. [Make sure that your argument applies to all z with $|z| = R$.]

[15 marks]

9. (a) Sketch the annulus $\{z \in \mathbf{C} : 4 < |z-2| < 7\}$, and mark the poles of the function

$$f(z) = \frac{11}{z^2 - z - 30}$$

on your sketch.

(b) Find the Laurent expansion of $f(z)$ valid in the above annulus.

(c) Determine whether this expansion converges at $z = -2$.

[15 marks]

10. Sketch the path $\gamma_R : [0, \pi] \to \mathbf{C}$ defined by $\gamma_R(t) = Re^{it}$, where $R > 0$.

Let a be a positive constant. Prove that

$$\int_{\gamma_R} \frac{e^{iaz}}{(z^2+1)^2}\,dz \to 0 \quad \text{as} \quad R \to \infty\,.$$

By integrating $e^{iaz}/(z^2+1)^2$ along a suitable contour, find

$$\int_0^\infty \frac{\cos(ax)}{(x^2+1)^2}\,dx\,.$$

[15 marks]

11. Find the principal value of the integral

$$\int_0^\infty \frac{x \sin 5x}{x^4 - 81}\,dx$$

by integrating an appropriate holomorphic function round a large semicircle in the upper half-plane, indented both at 3 and -3.

State without proof any results you use on the limiting values of the integrals round the semicircular parts of the contour.

[15 marks]