SECTION A

1. State (without proof) whether or not each of the following sequences (x_n) is i) increasing; ii) decreasing; iii) bounded above; iv) bounded below. State also the supremum, infimum, maximum, and minimum of each sequence for which they exist. (For each sequence you should state explicitly whether or not it has each of the properties i), ii), iii), and iv) above; and for each of the supremum, infimum, maximum, and minimum, you should either give its value or state explicitly that it doesn't exist.)

a) $x_n = 1 - \frac{1}{n} \quad (n \geq 1)$.

b) $x_n = (-1)^n \sqrt{n} \quad (n \geq 0)$. [8 marks]

2. Calculate the value of, and the first four convergents to, each of the following continued fractions:

a) $[1, 1, 1, 1, 1, 1, \ldots]$.

b) $[1, 2, 1, 2, 1, 2, \ldots]$.

(Recall the formulae: $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n p_{n-1} + p_{n-2}$ for $n \geq 2$; $q_0 = 1$, $q_1 = a_1$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 2$.)

[8 marks]

3. State without proof whether each of the following sets is open, closed, both, or neither.

a) $(-1, 1)$, as a subset of $\mathbf{R}$.

b) $\{(x, y) : -1 < x < 1\}$, as a subset of $\mathbf{R}^2$.

c) $\{(x, y) : 2 \leq x^2 + y^2 \leq 3\}$, as a subset of $\mathbf{R}^2$.

d) $\mathbf{R}^2$, as a subset of $\mathbf{R}^2$.

[8 marks]

4. A map $f\colon [0, 1] \to [0, 1]$ has two fixed points, one period 2 orbit, two period 3 orbits, and three period 4 orbits. How many fixed points does f^n have for each of $n = 2$, $n = 3$, and $n = 4$?

[6 marks]

5. Order the numbers 7, 20, 23, 40, 52, 64, 82 using the Sharkovsky order. (Your answer should be given in the form

$$40 \triangleright 20 \triangleright 7 \triangleright 82 \triangleright 64 \triangleright 52 \triangleright 23,$$

although this is not, of course, the correct ordering.)

[7 marks]

6. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2 - 2$. Determine the fixed and period 2 points of f.

[8 marks]

7. Calculate the Fourier Series expansion of t $(t \in [-\pi, \pi))$. [10 marks]

SECTION B

8.

a) Let (x_n) be the sequence defined by $x_0 = 3$ and

$$x_{n+1} = \frac{x_n}{2} + \frac{3}{x_n}$$

for each $n \geq 0$. Prove that (x_n) is a decreasing sequence which tends to $\sqrt{6}$ as $n \to \infty$. (You may use any results from the lectures without proof, provided that you state them clearly.) [9 marks]

b) Show that $|x_n - \sqrt{6}| < (1/6)^n$. How large should n be to ensure that x_n agrees with $\sqrt{6}$ to 1000 decimal places? [6 marks]

9.

a) What does it mean for an infinite set S to be countable? [3 marks]

b) Show that $\mathbf{Q}$ is countable. [4 marks]

c) Show that $\mathbf{R}$ is uncountable. [4 marks]

d) Show that if S and T are countable infinite sets, then so is their union $S \cup T$. [4 marks]

10.

a) State Sharkovsky's theorem (concerning the possible sets of periods of continuous maps $f: [0,1] \to [0,1]$). [4 marks]

b) Determine the Markov graph of the period 7 pattern $(1\,4\,5\,3\,6\,2\,7)$. Suppose that $f: [0,1] \to [0,1]$ is a continuous map with a periodic orbit of this pattern: what other periods of orbits must f have?

[11 marks]

11.

a) What does it mean for a square matrix P to be a stochastic matrix? Under what conditions is it true that there is a unique vector $\mathbf{x}$ such that $P^n \mathbf{x}_0 \to \mathbf{x}$ as $n \to \infty$ for all probability vectors $\mathbf{x}_0$? Determine this vector $\mathbf{x}$ for the matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

[10 marks]

b) Show that the map $f: [1, \infty) \to [1, \infty)$ defined by

$$f(x) = x + \frac{1}{x}$$

satisfies $|f(x) - f(y)| < |x - y|$ for all $x \neq y$, but has no fixed points. Why does this not contradict the contraction mapping theorem? [5 marks]

12.

a) What does it mean for a function $f\colon \mathbf{R} \to \mathbf{R}$ to be even? Explain why an even periodic function has no sine terms in its Fourier series expansion.

[3 marks]

b) Calculate the Fourier series expansion of $|\sin t|$. [7 marks]

c) Define what it means for a series

$$\sum_{r=1}^{\infty} f_r(t)$$

of functions $f_r\colon \mathbf{R} \to \mathbf{R}$ to converge *pointwise*, and to converge *uniformly*, to a function $f(t)$. What function does the Fourier series expansion in part b) converge to pointwise? Is the convergence uniform?

[5 marks]

13. Calculate the Fourier Series expansion of $|t|$ $(t \in [-\pi, \pi))$. By applying Parseval's theorem, evaluate

$$\sum_{r \text{ odd},\, r\geq 1}^{\infty} \frac{1}{r^4}.$$

[15 marks]