MATH241 Exam September 2000, Solutions

1.

a) Neither increasing nor decreasing. Bounded above and below. Supremum and maximum are 2, Infimum and minimum are 0.

b) Increasing, not decreasing. Bounded above and below. Infimum and minimum are 0, supremum is 2, no maximum.

2. $a_0 = 2$, $a_1 = 1$, $a_2 = 2$, $a_3 = 1$.

The formulae for the convergents p_n/q_n give $p_0 = 2$, $p_1 = 3$, $p_2 = 8$, and $p_3 = 11$; and $q_0 = 1$, $q_1 = 1$, $q_2 = 3$, and $q_3 = 4$. Thus the first four convergents are $2/1$, $3/1$, $8/3$, and $11/4$.

3. a) Open; b) Neither; c) Neither.

4. f^4 has $2^4 = 16$ fixed points, of which $2^2 = 4$ are also fixed points of f^2, and hence are not period 4 points. Thus f has 12 period 4 points, which constitute $12/4 = 3$ period 4 orbits.

5. The transition matrix is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix},$$

where the first row and column correspond to attending, and the second to missing.

The long term proportion of time spent in each of the two states is given by a probability column eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ of this matrix. Thus

$$\begin{aligned} x/2 + y/3 &= x \\ x + y &= 1, \end{aligned}$$

with solutions $x = 2/5$ and $y = 3/5$. Thus she attends $2/5$ of her lectures in the long run.

6. The fixed points are given by $f(x) = x$, or $x^3 - 3x^2 + 2x = 0$, with solutions $x = 0$, $x = 1$, and $x = 2$.

The stabilities can be determined by evaluating $f'(x) = 3x^2 - 6x + 3$ at each fixed point: we have $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = 3$. Hence $x = 1$ is a stable fixed point, the other two are unstable.

7. $|t|$ is an even function, so its Fourier series expansion is of the form

$$a_0/2 + \sum_{r=1}^{\infty} a_r \cos rt,$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\,dt = \frac{2}{\pi}\int_{0}^{\pi} t\,dt = \pi,$$

and

$$a_r = \frac{1}{\pi}\int_{-\pi}^{\pi} |t| \cos rt\,dt = \frac{2}{\pi}\int_{0}^{\pi} t \cos rt\,dt = -\frac{2}{r\pi}\int_{0}^{\pi} \sin rt\,dt = -\frac{2}{r^2\pi}(1-(-1)^r)$$

for $r \geq 1$. Hence the Fourier series expansion is

$$\pi/2 + \sum_{r=1}^{\infty} \frac{2((-1)^r - 1)}{r^2\pi} \cos rt.$$

8. The coefficients of the Fourier expansion of t^2 are $a_0 = 2\pi^2/3$, $a_r = 4(-1)^r/r^2$ for $r \geq 1$, and $b_r = 0$ for all r. Parseval's theorem states that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)^2\,dt = \frac{a_0^2}{4} + \frac{1}{2}\sum_{r=1}^{\infty}(a_r^2 + b_r^2).$$

In this case, we have

$$\int_{-\pi}^{\pi} f(t)^2\,dt = \int_{-\pi}^{\pi} t^4\,dt = 2\frac{\pi^5}{5},$$

so Parseval's theorem gives

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + \sum_{r=1}^{\infty}\frac{8}{r^4},$$

Or

$$\sum_{r=1}^{\infty}\frac{1}{r^4} = \frac{\pi^4}{8}\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{\pi^4}{90}.$$

9. We use the following theorem from lectures: Suppose that $f\colon [a,b] \to \mathbf{R}$ is an increasing function, that a sequence (x_n) is defined iteratively using f from some starting value $x_0 \in [a,b]$, and that $x_1 \leq x_0$. If there is a fixed point of f in $[a, x_0]$, then (x_n) is an decreasing sequence which tends to a limit l, which is the largest fixed point of f in $[a, x_0]$.

In this example $f(x) = x/2 + 7/(2x)$, so the fixed points of f are $\pm\sqrt{7}$. Also $f'(x) = 1/2 - 7/(2x^2)$, which is non-negative for $x \geq \sqrt{7}$, so we work in the interval $[a, b] = [\sqrt{7}, 3]$. Since $x_1 = 1 + 3/2 + 7/6 < x_0$, it follows from the theorem that (x_n) is decreasing and tends to $\sqrt{7}$ as $n \to \infty$.

The completeness axiom states that any non-empty subset of $\mathbf{R}$ which is bounded above has a least upper bound (and any non-empty subset of $\mathbf{R}$ which is bounded below has a greatest lower bound.

Let (x_n) be an decreasing sequence which is bounded below: by the completeness axiom, it has a greatest lower bound l (so $x_n \geq l$ for all n). Given any $\epsilon > 0$, there is some N such that $l \leq x_N < l + \epsilon$ (otherwise $l + \epsilon$ would be a lower bound), and since (x_n) is decreasing this means that $l \leq x_n < l + \epsilon$ for all $n \geq N$. Thus $x_n \to l$ as $n \to \infty$.

10. S is countable if there is a one-to-one correspondence between it and $\mathbf{N}$: or equivalently if there is a sequence (x_n) which includes all of the elements of S.

To show that $\mathbf{Q}$ is countable, define a sequence (y_n) by $y_{j(j+1)/2+k} = (1+k)/(1+j-k)$ for $j \geq 0$ and $0 \leq k \leq j$. This sequence includes all positive rationals, since if $p, q > 0$ then $y_n = p/q$ when $n = (p^2+2pq+q^2-p-3q-2)/2$. Hence the sequence $(x_n) = (0, y_0, -y_0, y_1, -y_1, \ldots)$ includes every element of $\mathbf{Q}$.

To show that $\mathbf{R}$ is uncountable, it is enough to show that the set S of real numbers in $[0, 1]$ whose decimal expansions only involve the digits 5 or 6 is uncountable. Let (x_n) $(n \geq 1)$ be any sequence of elements of S, and let the ith decimal digit of x_n be d_n^i. Then

$$x = \sum_{i=1}^{\infty} \frac{11 - d_i^i}{10^i}$$

is an element of S which is not included in the sequence. Hence S is uncountable.

The proof that the set S in part c) is countable is similar to the proof that $\mathbf{Q}$ is countable: the set A_k of points (x, y, z) in S with $x + y + z = k$ is finite, and a sequence (x_n) which includes all elements of S can be constructed by listing in turn the elements of A_3, A_4, A_5, and so on.

11. Let $\triangleright$ be the order on the positive integers given by

$$3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright 11 \triangleright \ldots$$

$$6 \rhd 10 \rhd 14 \rhd 18 \rhd 22 \rhd \dots$$

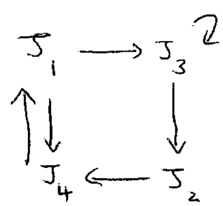
$$12 \rhd 20 \rhd 28 \rhd 36 \rhd 44 \rhd \dots$$

$$\dots$$

$$\dots \rhd 16 \rhd 8 \rhd 4 \rhd 2 \rhd 1.$$

If $f: [0,1] \to [0,1]$ is continuous and has a periodic orbit of period n, then it also has periodic orbits of period m for all m with $n \rhd m$.

The Markov graph of $(1\,3\,4\,2\,5)$ is



Since there are no primitive loops of length 3 in the Markov graph, it is possible to have a map $f: [0,1] \to [0,1]$ which has a periodic orbit of the given pattern, but no period 3 orbit. By Sharkovsky's theorem, all periods other than 3 must be present.

If $\pi \in S_5$ is a pattern of a periodic orbit of a reverse unimodal map, then it is a cyclic permutation which is reverse unimodal: i.e. it is decreasing on $\{1, 2, \dots k\}$ and increasing on $\{k, k+1, \dots, 5\}$ for some $k \in \{1, 2, 3, 4, 5\}$.

Let $\pi \in S_5$ be a reverse unimodal permutation.

If $k = 1$ or $k = 5$ then π cannot be cyclic (in the former case π is the identity, and in the latter $\pi(1) = 5$ and $\pi(5) = 1$.

Clearly either $\pi(1) = 5$ or $\pi(5) = 5$, and hence if π is cyclic then $\pi(1) = 5$.

If $k = 2$ then $\pi(2) = 1$ and hence $\pi = (1\,5\,4\,3\,2)$.

If $k = 3$ then $\pi(3) = 1$ and π is determined by $\pi(2)$: inspecting the three possibilities, only $(1\,5\,4\,2\,3)$ is cyclic.

If $k = 4$ then $\pi(4) = 1$ and π is determined by $\pi(5)$: inspecting the three possibilities, only $(1\,5\,3\,2\,4)$ is cyclic

Thus the three period 5 patterns which can arise from a unimodal map are $(1\,5\,4\,3\,2)$, $(1\,5\,4\,2\,3)$, and $(1\,5\,3\,2\,4)$.

12. A period n orbit P of $f: \mathbf{R} \to \mathbf{R}$ is stable if each $x \in P$ has a neighbourhood N such that $f^{kn}(y) \to x$ as $k \to \infty$ for all $y \in N$.

It is unstable if each $x \in P$ has a neighbourhood N such that $|f^n(y) - x| > |y - x|$ for all $y \in N$ with $y \neq x$.

The multiplier is defined to be $(f^n)'(x)$, where x is any point of P: equivalently, it is $f'(x_1) f'(x_2) \dots f'(x_n)$, where $P = \{x_1, \dots, x_n\}$.

Let $f_r(x) = r - x^2$. The fixed points of f_r are solutions of $f_r(x) = x$, namely

$$x = \frac{-1 \pm \sqrt{1+4r}}{2}$$

(which exist when $r \geq -1/4$). The fixed points of f_r^2 are solutions of $f_r(f_r(x)) = x$, i.e.

$$r - (r - x^2)^2 = x,$$

or

$$(x^2 + x - r)(x^2 - x + (1 - r)) = 0$$

(using the known factor x^2+x-r to factorize the quartic). Hence the period 2 points of f_r are the solutions of $x^2 - x + 1 - r = 0$, namely

$$x = \frac{1 \pm \sqrt{4r-3}}{2}.$$

Thus r has no period 2 orbits for $r \leq 3/4$, and a single period 2 orbit when $r > 3/4$.

A periodic orbit is stable when its multiplier lies in $(-1, 1)$. The multiplier of the period 2 orbit of f_r (when $r > 3$) is

$$f_r'\left(\frac{1+\sqrt{4r-3}}{2}\right) f_r'\left(\frac{1-\sqrt{4r-3}}{2}\right),$$

which, since $f_r'(x) = -2x$, is equal to

$$(1+\sqrt{4r-3})(1-\sqrt{4r-3}) = 1 - (4r-3) = 4(1-r).$$

Thus the period 2 orbit is stable provided $-1 < 4(1-r) < 1$, i.e. $3/4 < r < 5/4$. Thus the period 2 orbit is stable for $3/4 < r < 5/4$.

13.

a) t is an odd function, so its Fourier series expansion consists of sine terms only. The coefficients are given by

$$b_r = \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin rt\, dt = \frac{1}{\pi}\left[\frac{t\cos rt}{r}\right]_{\pi}^{-\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos rt}{r}\, dt = \frac{2(-1)^{r+1}}{r}.$$

Hence the Fourier series expansion is

$$\sum_{r=1}^{\infty} \frac{2(-1)^{r+1}}{r} \sin rt.$$

b) Integrating the Fourier series expansion of $t^4 - \pi^4/5$ term by term from 0 to u, we find that the Fourier series expansion of $u^5/5 - (\pi^4/5)u$ is

$$8 \sum_{r=1}^{\infty} \frac{(-1)^r (\pi^2 r^2 - 6)}{r^5} \sin ru,$$

and, replacing u with t and substituting the Fourier series expansion for t obtained in part a), it follows that the Fourier series expansion for t^5 is

$$5 \left(\frac{\pi^4}{5} \sum_{r=1}^{\infty} \frac{2(-1)^{r+1}}{r} \sin rt + 8 \sum_{r=1}^{\infty} \frac{(-1)^r (\pi^2 r^2 - 6)}{r^5} \sin rt, \right)$$

or

$$\sum_{r=1}^{\infty} \left(\frac{2(-1)^{r+1} \pi^4}{r} + \frac{40(-1)^r (\pi^2 r^2 - 6)}{r^5} \right) \sin rt.$$

c) In general, term by term differentiation of a the Fourier series expansion for $f(t)$ yields the Fourier series expansion for $f'(t)$ provided that (the periodic extension of) $f(t)$ is continuous and piecewise continuously differentiable.