SECTION A

1. State (without proof) whether or not each of the following sequences (x_n) is i) increasing; ii) decreasing; iii) bounded above; iv) bounded below. State also the supremum, infimum, maximum, and minimum of each sequence for which they exist. (For each sequence you should state explicitly whether or not it has each of the properties i), ii), iii), and iv) above; and for each of the supremum, infimum, maximum, and minimum, you should either give its value or state explicitly that it doesn't exist.)

a) $x_n = 1 + (-1)^n$ $(n \geq 0)$.

b) $x_n = \frac{2n}{n+1}$ $(n \geq 0)$.

[8 marks]

2. Let the continued fraction expansion of $e = 2.718281828\ldots$ be given by $[a_0, a_1, a_2, a_3, \ldots]$. Using your calculator, determine a_n for $0 \leq n \leq 3$ (you do not need to write anything down other than the values of each a_n). Hence calculate the first 4 convergents to e. (Recall the formulae: $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n p_{n-1} + p_{n-2}$ for $n \geq 2$; $q_0 = 1$, $q_1 = a_1$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 2$).

[7 marks]

3. State without proof whether each of the following sets is open, closed, both, or neither.

a) $(0, 2)$, as a subset of $\mathbf{R}$.

b) $\{(0, y) : y \in (0, 1)\}$, as a subset of $\mathbf{R}^2$.

c) $\{(x, y) : 1 < x^2 + y^2 \leq 2\}$, as a subset of $\mathbf{R}^2$.

[6 marks]

4. A map $f \colon [0, 1] \to [0, 1]$ is such that f^n has 2^n fixed points for each $n \geq 1$. How many period 4 orbits does f have?

[4 marks]

5. A student thinks that she can pass her exams by attending only some of her lectures. Her strategy is as follows: if she attends a given lecture, then she attends the following one with probability 1/2; if she misses a given lecture, then she misses the following one with probability 2/3.

Write down the matrix of transition probabilities which governs her attendance. In the long run, what proportion of lectures does she attend?

[6 marks]

6. Let $f\colon \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^3 - 3x^2 + 3x$. Find the fixed points of $f(x)$, and state whether each is stable or unstable.

[6 marks]

7. Calculate the Fourier series expansion of $|t|$ $(t \in [-\pi, \pi))$.

[10 marks]

8. The Fourier series expansion of t^2 $(t \in [-\pi, \pi))$ is

$$\frac{\pi^2}{3} + \sum_{r=1}^{\infty} \frac{4(-1)^r}{r^2} \cos rt.$$

By applying Parseval's theorem, show that

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

[8 marks]

SECTION B

9. Let (x_n) be the sequence defined by $x_0 = 3$ and

$$x_{n+1} = \frac{x_n}{2} + \frac{7}{2x_n}$$

for each $n \geq 0$. Prove that (x_n) is an decreasing sequence which tends to $\sqrt{7}$ as $n \to \infty$. (You may use any results from the lectures without proof, provided that you state them clearly).

State the completeness axiom of the real numbers, and show that any decreasing sequence of real numbers which is bounded below converges.

[15 marks]

10. What does it mean for an infinite set S to be countable?

a) Show that $\mathbf{Q}$ is countable.

b) Show that $\mathbf{R}$ is uncountable.

c) Show that the set S of all points $(x, y, z) \in \mathbf{R}^3$ such that x, y, and z are all positive integers is countable.

[15 marks]

11. State Sharkovsky's theorem (concerning the possible sets of periods of continuous maps $f\colon [0,1] \to [0,1]$).

Determine the Markov graph of the period 5 pattern $(1\,3\,4\,2\,5)$. Suppose that $f\colon [0,1] \to [0,1]$ is a continuous map with a periodic orbit of this pattern: what other periods of orbits must f have?

A continuous map $f\colon [0,1] \to [0,1]$ is said to be *reverse unimodal* if it is decreasing on $[0, 1/2]$ and increasing on $[1/2, 1]$. Determine all of the period 5 patterns which can arise as the pattern of a periodic orbit of a reverse unimodal map. (Hint: there are three of them).

[15 marks]

12. What does it mean for a period n orbit of a continuous map $f: \mathbf{R} \to \mathbf{R}$ to be stable, and to be unstable? If f has a continuous derivative $f'(x)$, define the *multiplier* of a period n orbit of f.

For each real number r, let $f_r: \mathbf{R} \to \mathbf{R}$ be given by $f_r(x) = r - x^2$. Determine the values of r for which f_r has a period 2 orbit, and the values of r for which this period 2 orbit is stable. (You are not required to determine the stability of the period 2 orbit in the cases where the multiplier is $+1$ or -1).

[15 marks]

13.

a) Calculate the Fourier series expansion of t $(t \in [-\pi, \pi))$.

b) The Fourier series expansion of t^4 $(t \in [-\pi, \pi))$ is

$$\frac{\pi^4}{5} + 8 \sum_{r=1}^{\infty} \frac{(-1)^r(\pi^2 r^2 - 6)}{r^4} \cos rt.$$

Integrate this series term by term and use your result from part a) to determine the Fourier series expansion of t^5 $(t \in [-\pi, \pi))$.

c) Under what conditions is term by term differentiation of a Fourier series expansion valid?

[15 marks]