SECTION A

1. State (without proof) whether or not each of the following sequences (x_n) is i) increasing; ii) decreasing; iii) bounded above; iv) bounded below. State also the supremum, infimum, maximum, and minimum of each sequence for which they exist. (For each sequence you should state explicitly whether or not it has each of the properties i), ii), iii), and iv) above; and for each of the supremum, infimum, maximum, and minimum, you should either give its value or state explicitly that it doesn't exist.)

a) $x_n = 1 - n^2 \quad (n \geq 0)$.

b) $x_n = \frac{1}{n!} \quad (n \geq 1)$. [8 marks]

2. Calculate the value of, and the first four convergents to, each of the following continued fractions:

a) $[2, 2, 2, 2, 2, 2, \ldots]$.

b) $[3, 1, 3, 1, 3, 1, \ldots]$.

(Recall the formulae: $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n p_{n-1} + p_{n-2}$ for $n \geq 2$; $q_0 = 1$, $q_1 = a_1$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 2$.)

[8 marks]

3. State without proof whether each of the following sets is open, closed, both, or neither.

a) $[-3, 3]$, as a subset of $\mathbf{R}$.

b) $\{(x, y) : -3 \leq x \leq 3\}$, as a subset of $\mathbf{R}^2$.

c) $\{(x, y) : x^2 + y^2 > 3\}$, as a subset of $\mathbf{R}^2$.

d) The empty set, as a subset of $\mathbf{R}^2$.

[8 marks]

4. A map $f : [0, 1] \to [0, 1]$ has two fixed points, one period 2 orbit, two period 3 orbits, and one period 4 orbit. How many fixed points does f^n have for each of $n = 2$, $n = 3$, and $n = 4$?

[6 marks]

5. Order the numbers 5, 14, 28, 32, 33, 44, 48 using the Sharkovsky order. (Your answer should be given in the form

$$44 \triangleright 33 \triangleright 32 \triangleright 5 \triangleright 14 \triangleright 28 \triangleright 48,$$

although this is not, of course, the correct ordering.)

[7 marks]

6. Let $f\colon \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2 - x^2$. Determine the fixed and period 2 points of f.

[8 marks]

7. Calculate the Fourier Series expansion of $|t|$ $(t \in [-\pi, \pi))$. [10 marks]

SECTION B

8.

a) Let (x_n) be the sequence defined by $x_0 = 2$ and

$$x_{n+1} = \frac{x_n}{2} + \frac{3}{2x_n}$$

for each $n \geq 0$. Prove that (x_n) is a decreasing sequence which tends to $\sqrt{3}$ as $n \to \infty$. (You may use any results from the lectures without proof, provided that you state them clearly.) [9 marks]

b) Show that $|x_n - \sqrt{3}| < (1/8)^n$. How large should n be to ensure that x_n agrees with $\sqrt{3}$ to 1000 decimal places? [6 marks]

9.

a) What is meant by a subsequence of a sequence (x_n)? State the Bolzano-Weierstrass theorem concerning the existence of convergent subsequences of a real-valued sequence (x_n). [4 marks]

b) For each of the following, either give an example of a sequence (x_n) with the stated properties, or explain why no such sequence exists.

i) A bounded sequence with a convergent subsequence.

ii) An unbounded sequence with a convergent subsequence.

iii) A bounded sequence with no convergent subsequence.

iv) An unbounded sequence with no convergent subsequence.

[11 marks]

10.

a) State Sharkovsky's theorem (concerning the possible sets of periods of continuous maps $f\colon [0,1] \to [0,1]$). [4 marks]

b) Determine the Markov graph of the period 7 pattern $(1\,4\,6\,2\,5\,3\,7)$. Suppose that $f\colon [0,1] \to [0,1]$ is a continuous map with a periodic orbit of this pattern: what other periods of orbits must f have?

[11 marks]

11.

a) What does it mean for a square matrix P to be a stochastic matrix? Under what conditions is it true that there is a unique vector $\mathbf{x}$ such that $P^n\mathbf{x}_0 \to \mathbf{x}$ as $n \to \infty$ for all probability vectors $\mathbf{x}_0$? Determine this vector $\mathbf{x}$ for the matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{pmatrix}.$$

[10 marks]

b) State the contraction mapping theorem (concerning fixed points of functions $f\colon A \to A$, where A is a closed subset of $\mathbf{R}^n$). Give an example of a contraction map $f\colon (0,1) \to (0,1)$ which has no fixed points. [5 marks]

12.

a) What does it mean for a function $f: \mathbf{R} \to \mathbf{R}$ to be odd? Explain why an odd periodic function has no constant or cosine terms in its Fourier series expansion.

[3 marks]

b) Calculate the Fourier series expansion of the 2π-periodic function $f(t)$ defined for $t \in [-\pi, \pi)$ by

$$f(t) = \begin{cases} -1 & \text{if } -\pi \le t < 0 \\ 1 & \text{if } 0 \le t < \pi. \end{cases}$$

[7 marks]

c) Define what it means for a series

$$\sum_{r=1}^{\infty} F_r(t)$$

of functions $F_r: \mathbf{R} \to \mathbf{R}$ to converge *pointwise*, and to converge *uniformly*, to a function $F(t)$. What function does the Fourier series expansion in part b) converge to pointwise? Is the convergence uniform?

[5 marks]

13. Calculate the Fourier Series expansion of t^2 $(t \in [-\pi, \pi))$. By applying Parseval's theorem, evaluate

$$\sum_{r=1}^{\infty} \frac{1}{r^4}.$$

[15 marks]