SECTION A

1. State (without proof) whether or not each of the following sequences (x_n) is i) increasing; ii) decreasing; iii) bounded above; iv) bounded below. State also the supremum, infimum, maximum, and minimum of each sequence for which they exist. (For each sequence you should state explicitly whether or not it has each of the properties i), ii), iii), and iv) above; and for each of the supremum, infimum, maximum, and minimum, you should either give its value or state explicitly that it doesn't exist.)

a) $x_n = \frac{1+(-1)^n}{n+1}$ $(n \geq 0)$.

b) $x_n = n^2 - 3n$ $(n \geq 0)$. [8 marks]

2. Let the continued fraction expansion of $\pi = 3.14159265\ldots$ be given by $[a_0, a_1, a_2, a_3, \ldots]$. Using your calculator, determine a_n for $0 \leq n \leq 3$ (you do not need to write anything down other than the values of each a_n). Hence calculate the first 4 convergents to π. (Recall the formulae: $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n p_{n-1} + p_{n-2}$ for $n \geq 2$; $q_0 = 1$, $q_1 = a_1$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 2$).

[7 marks]

3. State without proof whether each of the following sets is open, closed, both, or neither.

a) $[0, 1)$, as a subset of $\mathbf{R}$.

b) $[0, \infty)$, as a subset of $\mathbf{R}$.

c) $\{(x, y) : 1 < x - y < 2\}$, as a subset of $\mathbf{R}^2$.

[6 marks]

4. A map $f\colon [0, 1] \to [0, 1]$ is such that f^n has 3^n fixed points for each $n \geq 1$. How many period 4 orbits does f have?

[4 marks]

5. A lecturer starts each lecture that he gives by saying either "Okay" or "Right then". If he starts a given lecture by saying "Okay", he will start the next lecture in the same way with probability $2/3$; if he starts a given lecture by saying "Right then", he will start the next lecture in the same way with probability $3/4$.

Write down the matrix of transition probabilities which governs his behaviour. In the long run, what proportion of lectures does he start by saying "Okay"?

[6 marks]

6. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 1 - x^2$. Determine the fixed and period 2 points of f.

[6 marks]

7. Let $f(t)$ be the 2π-periodic function defined for $t \in [-\pi, \pi)$ by

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases}$$

Calculate the Fourier series expansion of $f(t)$. [10 marks]

8. The Fourier series expansion of $|t|$ $(t \in [-\pi, \pi))$ is

$$\frac{\pi}{2} + \sum_{r=1}^{\infty} \frac{2((-1)^r - 1)}{r^2 \pi} \cos rt.$$

By applying Parseval's theorem, show that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}.$$

[8 marks]

SECTION B

9. Let (x_n) be the sequence defined by $x_0 = 2$ and

$$x_{n+1} = x_n + 1 - \frac{x_n^2}{5}.$$

for each $n \geq 0$. Prove that (x_n) is an increasing sequence which tends to $\sqrt{5}$ as $n \to \infty$. (You may use any results from the lectures without proof, provided that you state them clearly).

State the completeness axiom of the real numbers, and show that any increasing sequence of real numbers which is bounded above converges.

[15 marks]

10. What does it mean for an infinite set S to be countable?

a) Show that $\mathbf{Q}$ is countable.

b) Show that $\mathbf{R}$ is uncountable.

c) Show that the set S of all subsets of $\mathbf{N}$ is uncountable.

[15 marks]

11. State Sharkovsky's theorem (concerning the possible sets of periods of continuous maps $f\colon [0,1] \to [0,1]$).

Determine the Markov graph of the period 5 pattern $(1\,2\,4\,3\,5)$. Suppose that $f\colon [0,1] \to [0,1]$ is a continuous map with a periodic orbit of this pattern: what other periods of orbits must f have?

A continuous map $f\colon [0,1] \to [0,1]$ is said to be *unimodal* if it is increasing on $[0, 1/2]$ and decreasing on $[1/2, 1]$. Determine all of the period 5 patterns which can arise as the pattern of a periodic orbit of a unimodal map. (Hint: there are three of them).

[15 marks]

12.

a) What does it mean for a function $f: \mathbf{R} \to \mathbf{R}$ to be even? Explain why an even periodic function has no sine terms in its Fourier series expansion.

b) Calculate the Fourier series expansion of $|\sin t|$.

c) Define what it means for a series

$$\sum_{r=1}^{\infty} f_r(t)$$

of functions $f_r: \mathbf{R} \to \mathbf{R}$ to converge *pointwise*, and to converge *uniformly*, to a function $f(t)$. State whether or not the Fourier series expansion of $|\sin t|$ converges uniformly to $|\sin t|$.

[15 marks]

13. Calculate the complex version

$$\sum_{r=-\infty}^{\infty} c_r e^{irt}$$

of the Fourier series expansion of the function $f(t) = e^t \ (t \in [-\pi, \pi))$.

By applying Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)^2 \, dt = \sum_{r=-\infty}^{\infty} |c_r|^2,$$

show that

$$\sum_{r=-\infty}^{\infty} \frac{1}{1+r^2} = \pi \coth \pi.$$

(Hint: remember that $\sinh t$ is defined by

$$\sinh t = \frac{e^t - e^{-t}}{2},$$

and that $\sinh(2t) = 2 \sinh t \cosh t$ and $\coth t = \cosh t / \sinh t$.)

[15 marks]