PAPER CODE NO.
MATH228

THE UNIVERSITY
of LIVERPOOL

SUMMER 2005 EXAMINATIONS

Bachelor of Science: Year 2
Master of Mathematics: Year 2
Master of Physics: Year 2
No qualification aimed for: Year 1

MECHANICS AND RELATIVITY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. All answers to Section A and the best THREE questions from Section B will be taken into account. Section A carries 55% of the available marks. The marks shown against the sections indicate their relative weights.

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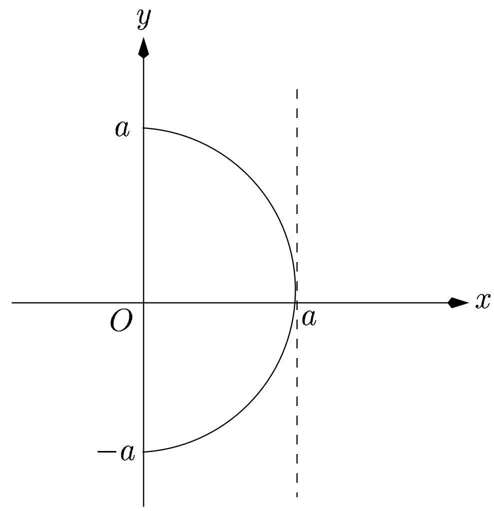
SECTION A

1. The diagram shows a uniform rigid semi-circular lamina of radius a and mass M in the $xy-$plane.



Find, by the use of polar coordinates or otherwise, the coordinates of the centre of mass of the lamina.

The lamina is now rotated through 2π about the line $x = a$. Show, by the use of the Theorem of Pappus-Guldin or otherwise, that the volume swept out is given by $(3\pi - 4)\pi a^3/3$.

[9 marks]

Hint:

$$\mathbf{r}_G = \frac{1}{A}\int_A \mathbf{r} dA.$$

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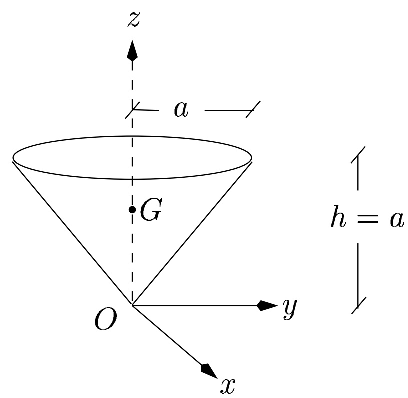
2. A uniform solid circular cone of mass M and height $h = a$ is placed symmetrically about the $z-$axis. The radius of its cross-section at $z = h$ is a. In terms of cylindrical polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$, the equation of the cone is given by $z = r$, where $0 \le z \le a,\ 0 \le \theta \le 2\pi$.



Given that the volume of the cone is $\pi a^3/3$ show that the moment of inertia of the cone about about the $x-$axis, i.e. I_{Ox}, is $3Ma^2/4$.

Write down the moment of inertia of the cone about the $y-$axis, i.e. I_{Oy}. Note that this is possible without any calculations.

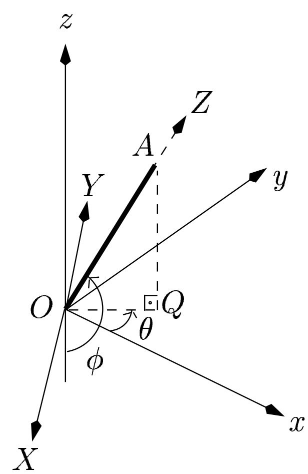
By the use of the Theorem of Parallel Axes or otherwise, deduce that the moment of inertia of the cone about an axis parallel to the $y-$axis and passing through its centre of mass $G(0, 0, 3a/4)$ is $3Ma^2/16$.

[11 marks]

Hint:

$$\text{Inertia matrix:}\quad \frac{M}{V}\int_V \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dV.$$

3. The diagram shows a uniform thin rigid rod OA. One end of the rod is attached to a fixed point O about which the rod swings freely under gravity. $Oxyz$ are fixed space axes with the $z-$axis vertically upwards and the $xy-$plane horizontal.



At time t, ϕ is the angle between the negative $z-$axis and OA as shown, and θ is the angle between the $x-$axis and OQ, where Q is the projection of A onto the $xy-$plane.

OX, OY, OZ, which form a right handed frame, are mutually perpendicular body axes, where OZ lies along OA, OX is in the plane AOQ, and OY is perpendicular to this plane.

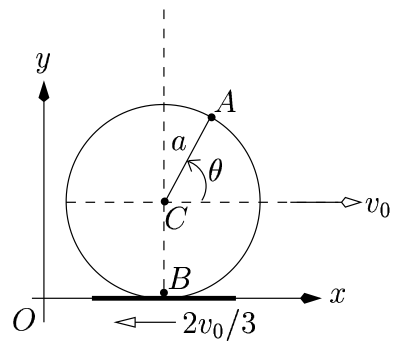
Show that, at time t the angular velocity $\boldsymbol{\omega}$ of the rod is given by:

$$\boldsymbol{\omega} = \dot{\phi}\,\mathbf{I} + \dot{\theta}\,\sin(\phi)\,\mathbf{J} - \dot{\theta}\,\cos(\phi)\,\mathbf{K},$$

where $\mathbf{I}, \mathbf{J}, \mathbf{K}$ are unit vectors along OX, OY, OZ respectively.

[7 marks]

4. A uniform disc of radius a rolls without slipping on the $xy-$plane in a straight line along a moving horizontal conveyer belt. Its centre C has a constant velocity v_0 in the positive $x-$direction whereas the velocity of the conveyer belt is $2v_0/3$ in the negative $x-$direction.



A is an arbitrary point on the circumference of the disc, θ is the angle that the line AC makes with the horizontal, and B is the instantaneous point of contact between the disc and the conveyer belt, as shown in the diagram.

Show that the magnitude of the velocity of the point A is given by

$$|\mathbf{v}_A| = \sqrt{2}\frac{v_0}{3}\sqrt{17 + 15\sin\theta}.$$

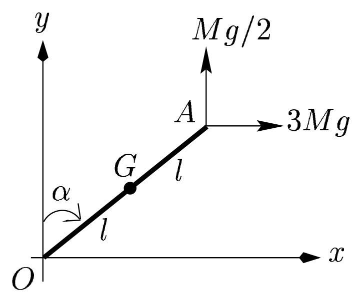
[11 marks]

Hint:

$$\mathbf{v}_P = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{CP},$$

where P is an arbitrary point on the disc.

5. A thin solid rod of mass M, length $2l$ with a uniform cross-section is to be lowered flat onto the ground from a vertical position. For this purpose, two ropes, attached to one end A of the rod, are used. As illustrated in the figure, one rope exerts a vertical force of $Mg/2$ whereas the other exerts a horizontal force of $3Mg$ at A. The other end of the rod is fixed at the point O, the origin of the $xy-$plane. The $x-$axis is chosen to be horizontal, the $y-$axis vertically upwards and the $z-$axis is perpendicular to the $xy-$plane.



The centre of mass G of the rod is at a distance l from the origin O and the moment of inertia of the rod about the axis Oz is I. Assuming that the rod remains in the $xy-$plane at all times and that at time t it makes an angle α with the vertical, show that:

$$I\ddot{\alpha} = 6Mgl\cos\alpha.$$

Multiplying the above equation of motion by $\dot{\alpha}$, integrating with respect to time t and using the condition that at $t = 0$ the rod is at rest vertically, show that the magnitude of its angular velocity is

$$\sqrt{\frac{12Mgl}{I}},$$

when the rod becomes horizontal.

[10 marks]

Hint:

$$\sum_i (\mathbf{OP}_i \times \mathbf{F}_i) = \dot{\mathbf{L}}_0.$$

6. Within the theory of special relativity, the $y-$ and $z-$axes of an inertial frame S are parallel to the corresponding axes of another inertial frame S'. The $x-$axes of S and S' are collinear and S' moves in the common $x-$direction with constant velocity v relative to S. The velocity of light in vacuo is c.

   A particle P travels along the common $x-$direction of S and S' with constant velocity u relative to S. Show that the velocity of P as observed from S' is given by:

$$u' = \frac{u - v}{1 - \dfrac{uv}{c^2}}.$$

   Deduce that if the velocity of light is measured independently in S and S', the same value c will be obtained.

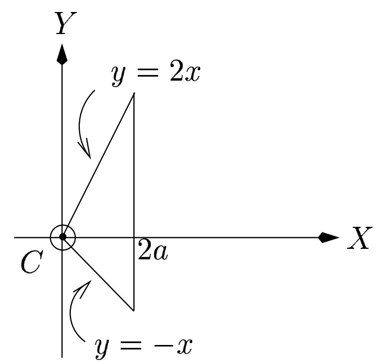
[7 marks]

Hint:

$$\Delta x' = \gamma(\Delta x - v\Delta t), \quad \Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}.$$

SECTION B

7. The diagram shows a uniform rigid lamina of mass M, bounded by the lines $y = 2x, y = -x$ and $x = 2a$. CX, CY are the body axes and the axis CZ is perpendicular to the $CXY-$plane. CX, CY and CZ form a right handed frame.



Show that the inertia matrix for the lamina, evaluated at C relative to these axes, is given by:

$$Ma^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Hence, find principal moments of inertia for the lamina at C.

[15 marks]

Hint: You can refer to the inertia matrix given in question 2.

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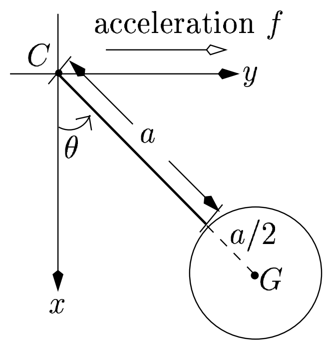
8. The diagram shows a pendulum consisting of a uniform circular disc of mass M and radius $a/2$ suspended from a smooth bearing at C using a thin rigid rod of length a and of negligible mass.

   The pendulum can swing freely under gravity in the $xy-$plane about the point C. The $x-$axis is chosen vertically downwards, the $y-$axis is horizontal and the $z-$axis is perpendicular to the $xy-$plane, forming a right handed frame.

   Using Routh's Rules and the Theorem of Parallel Axes, show that the moment of inertia of the pendulum about the axis Cz is $19Ma^2/8$.

   Assuming that C is moving with constant acceleration f in the positive $x-$direction, use the equation of motion involving the angular momentum of the pendulum about the moving point C to show that:

$$19a\ddot{\theta} = -12(f\cos\theta + g\sin\theta),$$

   where θ is the angle at time t between CG and the downward vertical through C, as shown in the diagram.

   Suppose now that C is accelerating with rate $f = g$ and the pendulum is in dynamical equilibrium with CG remaining at a constant angle $\theta = \alpha$ during the motion. Find the values of α for which this is possible.

[15 marks]

Hint:

$$\sum_i (\mathbf{CP}_i \times \mathbf{F}_i) = M(\mathbf{CG} \times \dot{\mathbf{v}}_C) + \dot{\mathbf{L}}_C.$$

9. At time t, the components of the angular velocity of a uniform axially symmetric rigid body, along the principal axes GX, GY, GZ at its centre of mass G, are $\omega_1(t), \omega_2(t)$ and $\omega_3(t)$, respectively. The corresponding principal moments of inertia at G about these axes are given as I_1, I_2 and I_3 respectively.

   Assuming that all external forces on the rigid body act at G and the body is symmetric about the principal axis GX, use the Euler form to show that ω_1 is a constant of the motion. Hence verify that

$$\omega_2(t) = c_1 \sin(\kappa t + c_2), \quad \omega_3(t) = c_1 \cos(\kappa t + c_2),$$

   where c_1 and c_2 are arbitrary constants and $\kappa = (I_2 - I_1)\omega_1/I_1$.

   Now show, by differentiating the magnitude of the angular velocity with respect to time or otherwise, that it too is a constant of the motion.

   A constant force F is now applied to this symmetric body at a point on the GX axis at a distance d from G; the direction of this constant force is parallel to GZ. Find a new expression for ω_3 as a function of time by considering the equations of motion of the angular momentum about G.

[15 marks]

Hint:

$$\sum_i (\mathbf{CP}_i \times \mathbf{F}_i) = M(\mathbf{CG} \times \dot{\mathbf{v}}_C) + \dot{\mathbf{L}}_C,$$

$$\begin{aligned}\dot{\mathbf{L}}_C &= [I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3]\mathbf{I} \\ &\quad + [I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1]\mathbf{J} \\ &\quad + [I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2]\mathbf{K}.\end{aligned}$$

10. A symmetric spinning top moves under gravity about a stationary fixed pivot O on its axis of symmetry. The centre of mass G of the top lies on this symmetry axis at a distance a from O.

Draw a clearly labelled diagram to define the conventional Euler angles θ, ϕ and ψ which specify the position of the top relative to fixed space axes $Oxyz$, where Oz is vertically upwards.

It may be assumed that the mass and the principal moments of inertia of the top are such that its motion at time t is given by the equations:

$$\begin{aligned} \dot{\psi} + \dot{\phi}\cos(\theta) &= s, \\ s\cos(\theta) + \dot{\phi}\sin^2(\theta) &= A, \\ \dot{\theta}^2 + \dot{\phi}^2\sin^2(\theta) + 2a\cos(\theta) &= B, \end{aligned}$$

where s, A and B are constants of motion, with $s \neq 0$.

At $t = 0$, the top is released from rest at the vertical position. Find the values of the constants A, B and hence show that during the motion which follows:

$$\dot{z}^2 = 2(z-a)^2(z-\nu) = 2F(z),$$

where $z = a\cos\theta$ and $\nu = \dfrac{s^2}{2} - a$.

Given that $\nu > a$, draw the graph of $F(z)$ for $-a \leq z \leq a$, and use it to deduce that the top will remain spinning with its axis of symmetry vertically upwards.

[15 marks]

11. Explain briefly, within the theory of special relativity, what is meant by an "inertial frame of reference".

    The $y-$ and $z-$axes of an inertial frame S are parallel to the corresponding axes of another inertial frame S'. The $x-$axes of S and S' are collinear and S' moves in the common $x-$direction with constant velocity v relative to S. The velocity of light in vacuo is c.

    (a) A thin straight rod lies along the $x-$axis of S' and is at rest relative to S'. An observer in S', measures the length of the rod as 10 metres. Assuming that $v = 2c/5$, in metres per second, use the Lorentz transformation to find the length of the rod, from the point of view of someone at rest in S.

    (b) Two events take place at different spatial locations in S but at the same spatial location relative to S'. Assuming that the time interval between the events as measured from S is t^*, find the corresponding time interval relative to S'.

[15 marks]

Hint: You can refer to the formulae given in question 6.