PAPER CODE NO.
MATH228

THE UNIVERSITY of LIVERPOOL

SUMMER 2004 EXAMINATIONS

Bachelor of Science: Year 2
Master of Mathematics: Year 2

MECHANICS AND RELATIVITY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. All answers to Section A and the best THREE questions from Section B will be taken into account. Section A carries 55% of the available marks. The marks shown against the sections indicate their relative weights.

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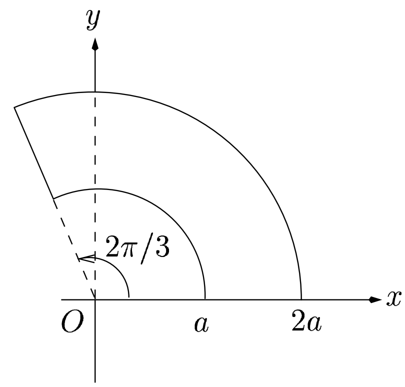
SECTION A

1. The diagram shows a uniform rigid lamina in the shape of a section of a circular ring lying in the xy-plane. The section is of angle $2\pi/3$ and radius r where $a \leq r \leq 2a$ as shown.
Show that the coordinates of the centre of mass G of this lamina are $(7\sqrt{3}a/6\pi, 7a/2\pi)$.
Now, the lamina is rotated through 2π about the x-axis. Using the Theorem of Pappus-Guldin, or otherwise, find the volume of this solid of revolution.

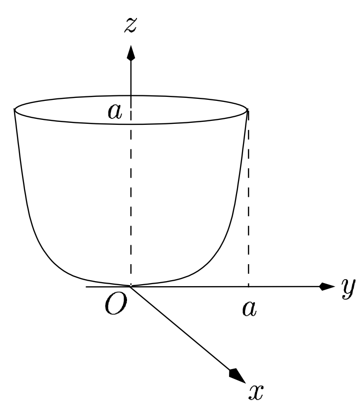


Hint: You may refer to

$$\mathbf{r}_G = \frac{1}{A}\int_A \mathbf{r}\,dA.$$

[8 marks]

2. A uniform solid of revolution of mass M and of height a is placed symmetrically about the z-axis. In terms of cylindrical polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$, the equation of this solid is given by $z = r^4/a^3$, where $0 \leq z \leq a,\ 0 \leq \theta \leq 2\pi$.
Explain briefly what you understand by "principal axes of inertia". Identify the principal axes of inertia for this solid at the origin O.
Now show that the volume of the solid is $2\pi a^3/3$ and hence deduce that the moment of inertia of the solid about the z-axis, i.e. I_{Oz}, is $3Ma^2/8$.



Hint: You may refer to

Inertia matrix: $$\frac{M}{V}\int_V \begin{pmatrix} y^2+z^2 & -\,xy & -\,xz \\ -xy & x^2+z^2 & -\,yz \\ -xz & -\,yz & x^2+y^2 \end{pmatrix} dV.$$

[8 marks]

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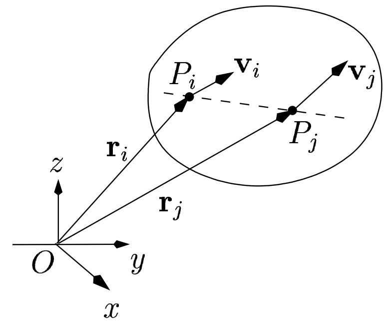
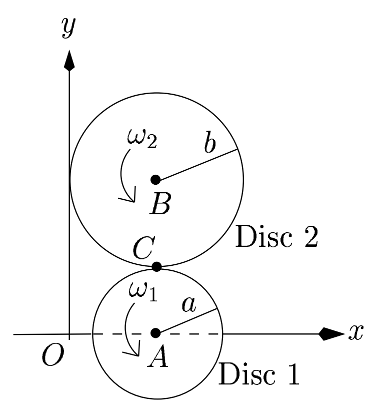
3. The diagram shows a uniform rigid body moving freely in space. Two arbitrary points P_i and P_j are chosen within the rigid body. Prove that the projection of the velocity $\mathbf{v}_i$ (that is, the velocity of the point P_i) onto the line connecting P_i to P_j is equal to the projection of $\mathbf{v}_j$ (the velocity of the point P_j) onto the same line.



[8 marks]

4. Two uniform rigid discs rotate without slipping as shown in the diagram. Disc 1, of radius a, is fixed at its centre A and has angular speed ω_1, whereas disc 2, of radius b, is also fixed at its centre B and has angular speed ω_2. The sense of angular velocity for both discs is shown in the diagram. Show that $\omega_1 = -(b/a)\omega_2$.

Hint: You may refer to $\mathbf{v}_P = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{CP}$



[8 marks]

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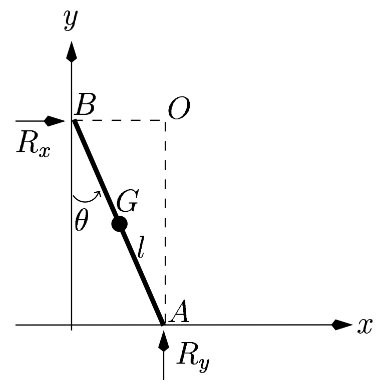
5. A rod of length l and mass M rests with one end A on the floor and the other end B in contact with the wall as shown in the diagram. Both the wall and the floor are polished so that the effect of friction is negligible. R_x and R_y are the reaction forces exerted on the rod by the wall and the floor, respectively. The rod is initially vertical. With a slight touch, the rod begins to slide downwards, both ends maintaining contact with the surfaces they initially were in contact with. At time t, θ is the angle between the wall and the rod, as shown in the diagram. Using the fact that the rod rotates about O, (where O is the instantaneous centre of rotation), show, by the use of the Theorem of Parallel Axes, that the moment of inertia of the rod about an axis perpendicular to the xy-plane through O, is $Ml^2/3$.



Next, using the conditions that when the rod is at rest $\theta = 0$ and $\dot{\theta} = 0$, deduce that the magnitude of the angular velocity is $\sqrt{3g[1 - \cos(\theta)]/l}$.

Hint: You may refer to the Theorem of Parallel Axes: $I_{\|} = I_G + Md^2$*, and to*

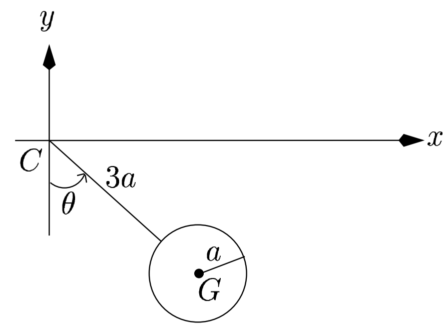
$$\sum_i \mathbf{OP}_i \times \mathbf{F}_i = \frac{d}{dt}\mathbf{L}_O,$$

[9 marks]

6. The diagram shows a pendulum consisting of a uniform circular disc of mass M and radius a suspended from a smooth bearing at C by a string of length $3a$ where the string is of negligible mass. The disc can swing freely under gravity about the point C, remaining in the xy-plane at all times. At time t, θ is the angle between the y-axis and the string as shown. Show, by applying the law of conservation of energy, that

$$\dot{\theta}^2(16Ma^2 + I) - 8Mg\cos\theta = \text{const.},$$

where I is the moment of inertia of the disc at G about the z-axis.



Hint: You may refer to

$$T = \frac{1}{2}M|\mathbf{v}_G|^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}_G.$$

[8 marks]

7. The corresponding coordinate axes of two inertial frames of reference S and S' are parallel to each other. S' is moving in the common x-direction with constant velocity v relative to S. Show that, within the theory of special relativity, if a thin straight rod of length L_0 lies along the x-axis of S' and is at rest relative to S', then its length relative to S is $L_0\sqrt{1-v^2/c^2}$.

Hint: You may refer to

$$\Delta x' = \gamma(\Delta x - v\Delta t), \quad \Delta t' = \gamma(\Delta t - \frac{v\Delta x}{c^2}) \quad \textit{where} \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

[6 marks]

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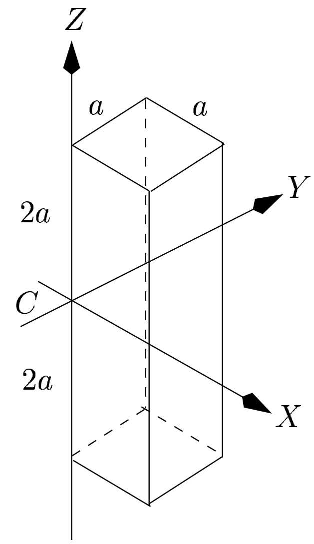
SECTION B



8. A uniform rigid rectangular parallelepiped of mass M and side lengths a, a, $4a$ is shown in the diagram. The origin C is located at the mid point of one of the longer sides and the axes $CXYZ$ are chosen such that they are lined up with the sides as shown. Show that the inertia matrix at C relative to $CXYZ$ is given by:

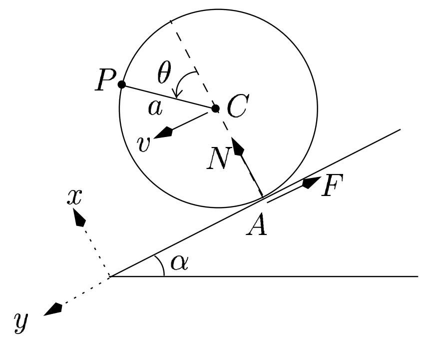
$$\frac{Ma^2}{12}\begin{pmatrix} 20 & -3 & 0 \\ -3 & 20 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Using this inertia matrix, find the Cartesian equations for the principal axes at C, show them on a diagram on the rectangular parallelepiped, and state the corresponding principal moments of inertia.

Hint: You may refer to the inertia matrix given in question 2.

[15 marks]

9. A rigid uniform disc of radius a and mass M is moving on a plane which is inclined at an angle α to the horizontal. The centre of the disc is denoted by C and the instantaneous point of contact with the plane is denoted by A, as shown. P is a fixed point on the circumference of the disc. The choice of the coordinate axes is as shown in the diagram. The plane of the disc remains in the vertical xy-plane at all times. At time t, θ is the angle between $\mathbf{CP}$ and the x-axis, and the centre C of the disc has velocity v in the positive y-direction.



The plane on which the disc is moving exerts a frictional force F and a reaction force N on the disc at A. At time $t = 0$, the disc is slipping over the inclined plane at the point of instantaneous contact with $v = 0$ and $\dot{\theta} = -s$, where s is strictly positive.

Use Newton's second law of motion to show that, so long as the disc continues to slip,

$$v = g(\sin\alpha - \mu\cos\alpha)t,$$

where μ is the coefficient of kinetic friction between the disc and the plane.

Given that the moment of inertia of the disc about Cz is I, use the equation for rotational motion about C to show that

$$a\dot{\theta} = (\frac{Mg\mu a^2}{I}\cos\alpha)t - as.$$

Now, considering the velocity of the disc at A and assuming that $\mu > \tan\alpha$, deduce that the disc stops slipping on the plane when $t = t_0$, where

$$t_0 = \frac{as}{g[\mu\cos\alpha(1 + \dfrac{Ma^2}{I}) - \sin\alpha]}.$$

Also explain why the disc starts rolling up the plane when $t = t_0$.

Hint: You may refer to

$$\sum_i \mathbf{F}_i = M\dot{\mathbf{v}}_G, \quad \textit{and} \quad \sum_i (\mathbf{CP}_i \times \mathbf{F}_i) = M(\mathbf{CG} \times \dot{\mathbf{v}}_C) + \dot{\mathbf{L}}_C.$$

[15 marks]

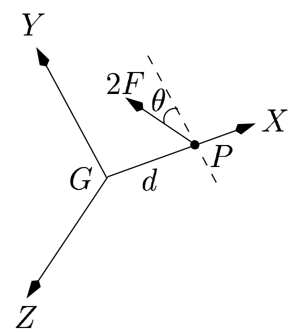
10. At time t, the components of the angular velocity of a uniform rigid body along the principal axes GX, GY, GZ at its centre of mass are $\omega_1(t), \omega_2(t), \omega_3(t)$ respectively. The corresponding principal moments of inertia are given to be $I_1 = 2I, I_2 = I_3 = I$ respectively.

If all the external forces on the body act at G, by using the Euler form, show that, at time t

$$\omega_1 = s, \quad \dot{\omega}_2 = -s\omega_3, \quad \dot{\omega}_3 = s\omega_2,$$

where s is constant. Hence, deduce that $\omega_2(t)$ and $\omega_3(t)$ satisfy a differential equation of the form: $\ddot{\omega}_k + s^2\omega_k = 0,\ k = 1, 2$ and find the representations for $\omega_2(t)$ and $\omega_3(t)$.

Now, consider the case when all the external forces on the body act at G, except a force of constant magnitude $2F$ in the XY-plane acting at a point P on GX at a distance d from G making a constant angle $\theta = \pi/3$ with the GY axis as shown in the diagram. Show that the component of the angular velocity $\omega_2(t)$ now can be found as



$$\omega_2(t) = a\cos(st + b) - \frac{Fd}{Is},$$

where s is a constant of the motion.

Hint: You may refer to

$$\sum_i (\mathbf{CP}_i \times \mathbf{F}_i) = M(\mathbf{CG} \times \dot{\mathbf{v}}_C) + \dot{\mathbf{L}}_C,$$

and

$$\dot{\mathbf{L}}_C = [I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3]\mathbf{I} + [I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1]\mathbf{J} + [I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2]\mathbf{K}.$$

[15 marks]

11. A symmetric spinning top moves under gravity about a stationary fixed pivot O on its axis of symmetry. The centre of mass G of the top lies on this symmetry axis at distance a from O.

Draw a clearly labelled diagram to define the conventional Euler angles θ, ϕ, ψ which specify the position of the top relative to fixed space axes $Oxyz$, where Oz is vertically upwards.

It may be assumed that the mass and the principal moments of inertia of the top are such that its motion at time t is given by the equations:

$$\begin{aligned} \dot{\psi} + \dot{\phi}\cos\theta &= 2s, \\ 2s\cos\theta + 2\dot{\phi}\sin^2\theta &= A, \\ \dot{\theta}^2 + \dot{\phi}^2\sin^2\theta + 2\mu a\cos\theta &= B, \end{aligned}$$

where s, A, B are constants of the motion, with $s \neq 0$, and μ is a physical constant.

At time $t = 0$, the top is released from a position where $\theta = \pi/3$ with $\dot{\theta} = 0$ and $\dot{\phi} = 0$. Find the values of the constants A and B and hence, given that $s^2 = 4\mu a$, show that during the ensuing motion:

$$\dot{z}^2 = \mu f(z), \quad \text{where} \;\; z = a\cos\theta, \;\; \text{and} \;\; f(z) = z(z - 2a)(2z - a).$$

Draw the graph of $f(z)$ for $-a \leq z \leq a$, and use it to deduce that the motion of the top will be confined between the horizontal planes $z = 0$ and $z = a/2$.

[15 marks]

12. Within the theory of special relativity, the y- and z-axes of an inertial frame S are parallel to the corresponding axes of another inertial frame S'. The x-axes of S and S' are collinear and S' moves in the common $x-$direction with constant velocity v relative to S. The velocity of light in vacuo is c.

(a) Two events which take place at different spatial locations in S are observed from S' to occur at the same time t' relative to S'. Show that the events cannot occur simultaneously with respect to S.

(b) Two events take place at different spatial locations in S but at the same spatial location relative to S'. Assuming that the time interval between the events as measured from S is Δt, show that the corresponding time interval relative to S' is $\Delta t\sqrt{1 - v^2/c^2}$.

(c) If a particle P travels along the common x-direction of S and S' with constant velocity u relative to S, then show that the velocity of P as observed from S' is given by:

$$u' = \frac{u - v}{1 - uv/c^2}.$$

Hint: You may refer to the formulae given in question 7.

[15 marks]