PAPER CODE NO.
MATH228

THE UNIVERSITY
of LIVERPOOL

SUMMER 2003 EXAMINATIONS

Bachelor of Science: Year 2
Master of Mathematics: Year 2

MECHANICS AND RELATIVITY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. All answers to Section A and the best THREE questions from Section B will be taken into account. Section A carries 55% of the available marks. The marks shown against the sections indicate their relative weights.

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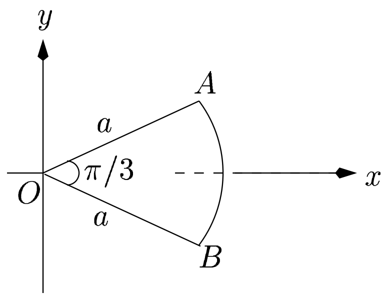
SECTION A

1. The diagram shows a uniform rigid lamina AOB of mass M in the $xy-$plane. The lamina is a sector of a disc of radius a, whose centre is at the origin O, with the angle AOB being $\pi/3$.



Find, by the use of polar coordinates or otherwise, the coordinates of the centre of mass of the lamina.

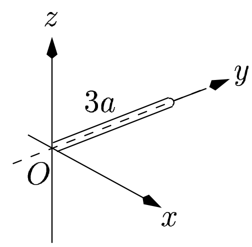
The lamina is now rotated through 2π about the $y-$axis. Show, by the use of the Theorem of Pappus-Guldin or otherwise, that the volume swept out is given by $2\pi a^3/3$.

[9 marks]

Hint:

$$\mathbf{r}_G = \frac{1}{A}\int_A \mathbf{r}\, dA.$$

2. A uniform rigid rod of mass M and length $3a$ lies along the $y-$axis with one end at the origin O, as shown in the diagram.



Assuming that the rod is of negligible thickness, show that its inertia matrix at the origin O, relative to the $x, y, z-$axes, is given by:

$$3Ma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the use of the Theorem of Parallel Axes or otherwise, deduce that the moment of inertia of the rod about an axis passing through its centre of mass G and parallel to the $x-$axis is $3Ma^2/4$.

[8 marks]

Hint:

$$\text{Inertia matrix:} \quad \frac{M}{V}\int_V \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dV.$$

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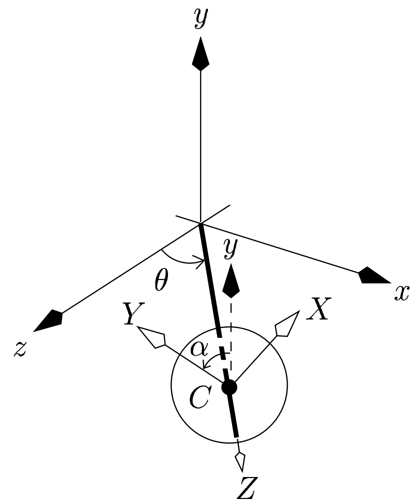
3. The diagram shows a uniform rigid thin rod rotating on the $xz-$plane. A uniform rigid circular disc is attached to the rod at its centre C, in such a way that it is perpendicular to the rod, and is rotating about C.



CX, CY and CZ are mutually perpendicular body axes, where CX and CY lie on the disc and CZ lies along the axis of the rod. At time t, θ is the angle between the $z-$axis and CZ, and α is the angle shown between upward vertical Cy and CY.

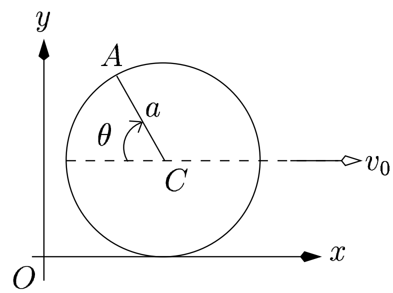
Show that at time t the angular velocity of the disc is given by

$$\boldsymbol{\omega} = \dot{\theta}\sin(\alpha)\mathbf{I} + \dot{\theta}\cos(\alpha)\mathbf{J} + \dot{\alpha}\mathbf{K},$$

where $\mathbf{I}, \mathbf{J}$ and $\mathbf{K}$ are unit vectors along CX, CY and CZ respectively.

[6 marks]

4. A uniform rigid disc of radius a rolls on the $xy-$plane without slipping, and its centre C has a constant velocity v_0 in the positive $x-$direction.



A is an arbitrary point on the circumference of the disc and θ is the angle that the line AC makes with the horizontal, as shown in the diagram. Show that the magnitude of velocity of the point A is given by

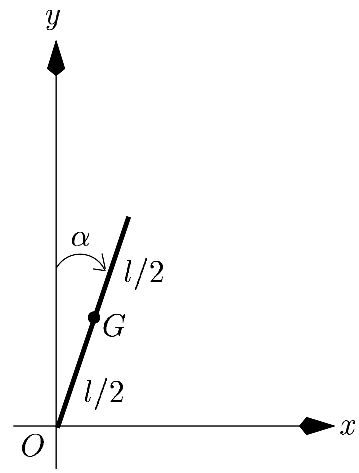
$$|\mathbf{v}_A| = v_0\sqrt{2[1+\sin(\theta)]}.$$

[10 marks]

Hint:

$$\mathbf{v}_A = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{CA}.$$

5. A thin solid chimney of mass M, length l and of uniform cross-section is to be demolished. With a small explosion at its base, fixed at the origin O, the chimney starts rotating in the $xy-$plane about O, under the acceleration of its own weight. The $x-$axis is chosen to be horizontal and the $y-$axis vertically upwards.



Calling the moment of inertia of the chimney about the axis Oz by I and assuming that the chimney remains in the $xy-$plane all the time and that at time t it makes the angle α with the vertical, show that

$$I\ddot{\alpha} = \frac{Mgl}{2}\sin(\alpha).$$

Integrating the above equation of motion with respect to time t and observing that at $t = 0$ the chimney is at rest vertically upwards, show that the angular velocity has a maximum value

$$\sqrt{\frac{Mgl}{I}}$$

just as the chimney hits the ground.

[9 marks]

Hint:

$$\sum_i (\mathbf{OP}_i \times \mathbf{F}_i) = \dot{\mathbf{L}}_0.$$

6. At time t, the components of the angular velocity of a uniform axially symmetric rigid body, along the principal axes GX, GY, GZ at its centre of mass G, are $\omega_1(t), \omega_2(t)$ and $\omega_3(t)$, respectively. The corresponding principal moments of inertia at G about these axes are given as $I_1 = I_2 = 2I$ and $I_3 = I$ respectively.

   Assuming that all external forces on the rigid body act at G, use the Euler form to show that ω_3 is a constant of the motion. Hence, verify that
$$\omega_1(t) = c_1 \sin\left(\frac{1}{2}\omega_3 t + c_2\right),$$
   where c_1 and c_2 are arbitrary constants.

   [7 marks]

   Hint:
$$\begin{aligned}\dot{\mathbf{L}}_C &= [I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3]\mathbf{I} + [I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1]\mathbf{J} \\ &\quad + [I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2]\mathbf{K}.\end{aligned}$$

7. Consider, within the theory of special relativity, two inertial frames of reference S and S'. Assume that the corresponding coordinate axes are parallel to each other and S' is moving in the common $x-$direction with constant velocity v relative to S.

   We observe two events: event 2 is observed from S to occur later than event 1. Assuming that the spatial separation of the events relative to S is less than the distance travelled by light in the time interval between the events, show that the events occur in the same chronological order relative to S'. State why this result is consistent with the Principle of Causality.

   [6 marks]

   Hint:
$$\Delta x' = \gamma(\Delta x - v\Delta t), \quad \Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \quad \textit{where} \quad \gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}.$$

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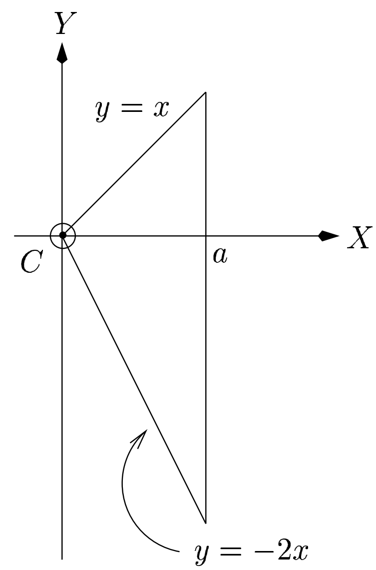
SECTION B

8. The diagram shows a uniform rigid lamina of mass M, bounded by the lines $y = x, y = -2x$ and $x = a$. CX, CY are the body axes and the axis CZ is perpendicular to the $CXY-$plane as shown.



Show that the inertia matrix for the lamina, evaluated at C relative to these axes, is given by:

$$\frac{Ma^2}{4}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

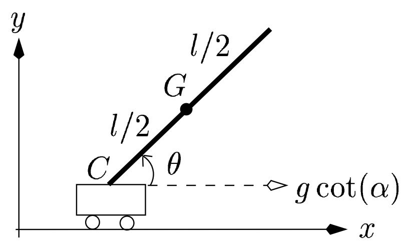
Hence, find principal moments of inertia for the lamina at C.

Assuming that the lamina is rotating about CY with angular velocity of magnitude ω, find the angular momentum vector $\mathbf{L}_C$ at C, in terms of the unit vectors $\mathbf{I}, \mathbf{J}, \mathbf{K}$ along CX, CY, CZ, respectively. Verify that the angle between $\mathbf{L}_C$ and CY is approximately 26.57^o.

[15 marks]

Hint: You can refer to the inertia matrix given in question 2.

9. A long rigid rod of mass M and of uniform cross-section is connected to the point C of a device which moves parallel to the horizontal $x-$axis with constant acceleration $g\cot(\alpha)$, $0<\alpha<\pi/2$, as shown in the diagram.



The length of the rod is l and at time t, CG makes an acute angle θ with the $x-$direction. Given that the motion takes place in the $xy-$plane, where the $y-$axis is vertically upwards, use the equations of motion of the angular momentum of the rod about the point C to show that

$$I\ddot{\theta}=\frac{Mgl}{2}\cot(\alpha)\sin(\theta)-\frac{Mgl}{2}\cos(\theta),$$

where I is its moment of inertia about the axis perpendicular to the $xy-$plane and going through C.

Deduce that it is possible to keep the rod in dynamical equilibrium with CG inclined at a constant angle α to the horizontal. Comment briefly on the situation for the case when $\alpha=\pi/2$.

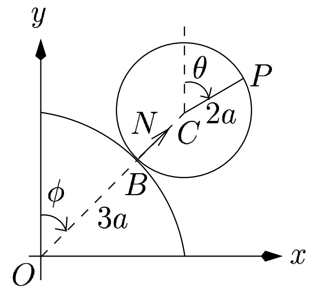
Let $\theta(t)=\alpha+\varepsilon(t)$, where $|\varepsilon|$ is small. Hence show that the above equilibrium is dynamically unstable.

[15 marks]

Hint:

$$\sum_i(\mathbf{CP}_i\times\mathbf{F}_i)=M(\mathbf{CG}\times\dot{\mathbf{v}}_C)+\dot{\mathbf{L}}_C.$$

10. The diagram shows a rigid uniform disc of mass M, centre C and radius $2a$ which rolls down the outside surface of a cylinder of radius $3a$, whose longitudinal axis passes through the origin O. The motion remains in the $xy-$plane at all times where the $x-$axis is horizontal and the $y-$axis is vertically upwards. At time t, ϕ is the angle between the upward vertical and OB, where B is the instantaneous point of contact between the disc and the cylinder, and θ is the angle between the upward vertical and CP, a body line on the disc.



Considering the equation of motion for the point C, show that

$$N = Mg\cos(\phi) - 5Ma\dot{\phi}^2,$$

where N is the normal component of the force at B, exerted by the cylinder on the disc.

Given that the moment of inertia of the disc about Cz, i.e. I_{Cz}, is $Ma^2/2$, apply the law of conservation of energy to obtain:

$$50a\dot{\phi}^2 + a\dot{\theta}^2 + 20g\cos(\phi) = \text{constant}.$$

At time t, the disc is projected from its highest position $\phi = 0$ with $\dot{\phi} = \Omega$, where Ω is constant. Assuming that the disc and the cylinder remain in contact and that no slipping occurs between them at B, show that

$$a\dot{\phi}^2 = a\Omega^2 + \frac{16}{45}g(\cos(\phi) - 1).$$

Hence, also show that if $\Omega \approx 0$, then the disc will lose contact with the cylinder when $\phi \approx 50.21^o$.

[15 marks]

Hint:

$$\sum_i \mathbf{F}_i = M\dot{\mathbf{v}}_G,$$

$$T_0 = \frac{1}{2}M|\mathbf{v}_G|^2 + T_G \quad \textit{where} \quad T_G = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}_G.$$

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11. A symmetric spinning top moves under gravity about a stationary fixed pivot O on its axis of symmetry. The centre of mass G of the top lies on this symmetry axis at a distance a from O.

Draw a clearly labelled diagram to define the conventional Euler angles θ, ϕ and ψ which specify the position of the top relative to fixed space axes $Oxyz$, where Oz is vertically upwards.

It may be assumed that the mass and the principal moments of inertia of the top are such that its motion at time t is given by the equations:

$$\dot{\psi} + \dot{\phi}\cos(\theta) = s, \tag{1}$$
$$s\cos(\theta) + 2\dot{\phi}\sin^2(\theta) = A, \tag{2}$$
$$\dot{\theta}^2 + \dot{\phi}^2\sin^2(\theta) + \mu a\cos(\theta) = B, \tag{3}$$

where μ is a physical constant and s, A and B are constants of motion, with $s \neq 0$.

Verify that the above equations of motion allow existence of a state of dynamical equilibrium in which the top precesses indefinitely, with OG rotating about Oz with constant angular velocity Ω and inclined at a constant angle α to Oz, where $0 < \alpha < \pi$.

Differentiate equations (2) and (3) with respect to time and combine your resulting equations with (1) to eliminate the terms involving $\ddot{\phi}$ and s. Hence, show that

$$2\ddot{\theta} = [\dot{\phi}^2\cos(\theta) - \dot{\phi}\dot{\psi} + \mu a]\sin(\theta).$$

Deduce that the above state of steady precession is possible only if:

$$\dot{\psi}^2 > 4a\mu\cos(\alpha).$$

[15 marks]

12. Explain briefly, within the theory of special relativity, what is meant by an "inertial frame of reference".

    The $y-$ and $z-$axes of an inertial frame S are parallel to the corresponding axes of another inertial frame S'. The $x-$axes of S and S' are collinear and S' moves in the common $x-$direction with constant velocity v relative to S. The velocity of light in vacuo is c.

    (a) A particle P travels along the common $x-$direction of S and S' with constant velocity u relative to S. Show that the velocity of P as observed from S' is given by:

    $$u' = \frac{u - v}{1 - \frac{uv}{c^2}}.$$

    Deduce that if the velocity of light is measured independently in S and S', the same value c will be obtained.

    (b) A thin straight rod lies along the $x-$axis of S' and is at rest relative to S'. An observer, moving in S', measures the length of the rod as 5 metres. Assuming that $v = c/4$, in metres per second, use the Lorentz transformation to show that, from the point of view of someone at rest in S, the length of the rod will appear to be approximately 4.84 metres.

[15 marks]

Hint: You can refer to the formulae given in question 7.