PAPER CODE NO. **MATH228**

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THE UNIVERSITY *of* LIVERPOOL

MAY EXAMINATIONS 2007

Bachelor of Arts: Year 2
Bachelor of Science: Year 2
Bachelor of Science: Year 3
Master of Mathematics: Year 2
Master of Mathematics: Year 3
Master of Physics: Year 2

CLASSICAL MECHANICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. All answers to Section A and the best THREE questions from Section B will be taken into account. Section A carries 55% of the available marks. The marks shown against the sections indicate their relative weights.

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SECTION A

1. A particle moves in space in accordance with the equation

$$x^3 + \cos(2y) - 3z = At,$$

where A is a constant.

Given that, at time $t = 0$, the position vector $\mathbf{r}$ and the velocity $\mathbf{v}$ of the particle are

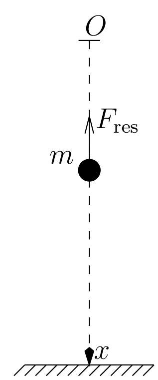
$$\mathbf{r} = B\,\mathbf{i} + 3\mathbf{k} \quad \text{and} \quad \mathbf{v} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k},$$

where B is a constant, find A and B. [8 marks]

2. A particle of mass m is dropped from the origin O with no initial speed in a medium with resistance. The coordinate axis x is chosen to be vertically downwards as shown in the Figure. The magnitude of the resisting force is assumed to be $F_{\text{res}} = mg\lambda v$, where g is the gravitational acceleration and λ is a constant. Show that the speed of the particle is given by

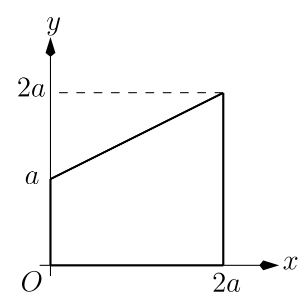
$$v(t) = v_l\,(1 - \mathrm{e}^{-gt/v_l}),$$

clarifying the notation for v_l, and hence find its displacement $x(t)$.



[10 marks]

3. A rigid uniform lamina lies in the xy−plane and is bounded by the lines $x = 2a, y = x/2 + a$ and the coordinate axes x and y, as shown in the Figure.
Find the coordinates of the centre of mass of this lamina.
The lamina is now rotated through 2π about the line $y = -2a/9$. Show, by the use of the Theorem of Pappus-Guldin or otherwise, that the volume swept out is given by $6\pi a^3$.



[9 marks]

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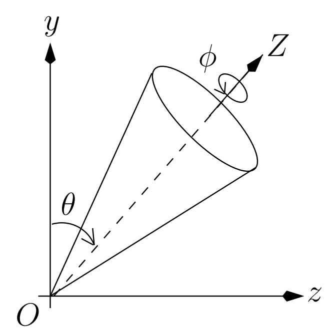
4. (a) The vector $\boldsymbol{\omega}$ is said to be the angular velocity vector of a rigid body, if $\dfrac{d\mathbf{a}}{dt} = \boldsymbol{\omega} \times \mathbf{a}$ holds true for every vector $\mathbf{a}$ embedded into the body. Prove that the vector $\boldsymbol{\omega}$ defined in this way is uniquely determined.

[3 marks]

(b) A rigid uniform circular cone is attached at its vertex to a stationary fixed pivot at the origin O. The motion of the cone about O is constrained so that its axis of symmetry OZ always remains in the $yz-$plane. At time t, θ is the angle between the vertically upwards $y-$axis and the cone's axis of symmetry OZ, and ϕ is the angle denoting the rotation about the $OZ-$axis.



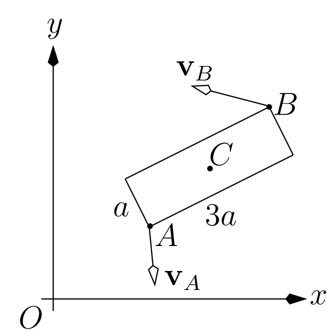
Assuming that $Oxyz$ form a right-handed frame, show that the angular velocity of the cone at time t is given by

$$\boldsymbol{\omega} = \dot{\theta}\,\mathbf{i} + \dot{\phi}\cos\theta\,\mathbf{j} + \dot{\phi}\sin\theta\,\mathbf{k},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote unit vectors along the axes x, y, z respectively.

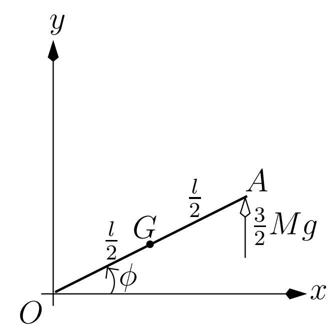
[8 marks]

5. A rigid rectangular lamina of sides a and $3a$ with centre C moves in the $xy-$plane at all times, as shown in the Figure. The velocities of the two corners A and B of the lamina are given to be $\mathbf{v}_A$ and $\mathbf{v}_B$ at time t, respectively. Find the velocity of the centre point C, i.e. $\mathbf{v}_C$, in terms of $\mathbf{v}_A$ and $\mathbf{v}_B$.
Also show that the magnitude of the angular velocity of the lamina is given by $|\mathbf{v}_A - \mathbf{v}_B|/(\sqrt{10}a)$.



[9 marks]

6. A thin solid rod of mass M, length l and of uniform cross section is being lifted from the ground to a vertical position by the use of a thrust applied at one end A of the rod, as shown in the Figure. The magnitude of the thrust is $3Mg/2$ and this thrust is in the vertically upwards $y-$direction. The other end of the rod is fixed at the point O, the origin of the $xy-$plane and the rod remains in the $xy-$plane at all times. The $z-$axis is oriented out of the page, so that $Oxyz$ form a right-handed frame.



Given that the moment of inertia of the rod about the $z-$axis is I and that, at time t, the rod makes an angle ϕ with the horizontal $x-$axis, using the equation for rotational motion about the origin O, show that

$$Mgl\cos\phi = I\ddot{\phi}.$$

Now, multiplying the above equation by $\dot{\phi}$, integrating with respect to time t and using the condition that at $t = 0$ the rod is at rest horizontally, show that the magnitude of its angular velocity is $\sqrt{2Mgl/I}$ when the rod is vertically upwards.

[8 marks]

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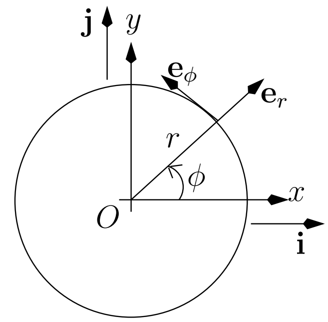
SECTION B

7. (a) Using the Figure on the right, find the unit vector $\mathbf{e}_r(t)$ in the radial direction and $\mathbf{e}_\phi(t)$ in the tangential direction, where t stands for time, in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$ along the coordinate axes x, y respectively.
Hence, deduce that



$$\dot{\mathbf{e}}_r = \dot{\phi}\,\mathbf{e}_\phi, \quad \dot{\mathbf{e}}_\phi = -\dot{\phi}\,\mathbf{e}_r,$$

and use these derivatives and the position vector

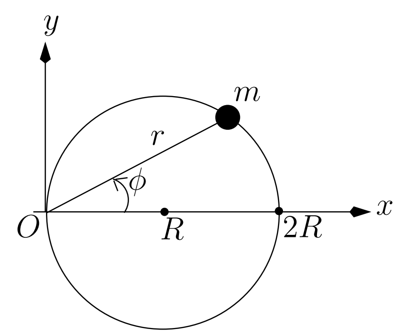
$$\mathbf{r}(t) = r(t)\,\mathbf{e}_r(t)$$

to find the components of the velocity vector $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.

[7 marks]

(b) By differentiating the velocity vector $\mathbf{v}$ with respect to time, the acceleration vector in polar coordinates can be found as

$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2)\,\mathbf{e}_r + (2\dot{r}\dot{\phi} + r\ddot{\phi})\,\mathbf{e}_\phi.$$



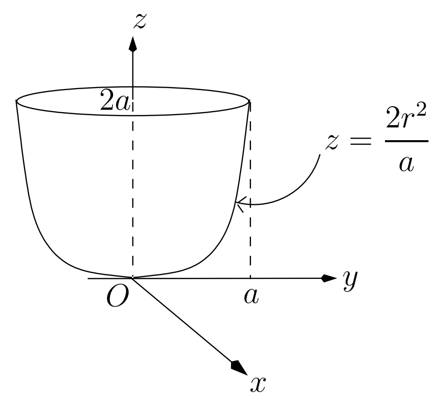
Consider a particle of mass m moving on the circle of centre $(R, 0)$ and radius R as shown in the Figure on the right. Given the equation of the circle is $r = 2R\cos\phi$, where r denotes the distance to the circle from the origin and ϕ is the angle between the horizontal $x-$axis and r, and that $a_\phi = 0$ throughout the motion of the particle, show that

$$\dot{\phi} = \frac{C}{r^2},$$

where C is an arbitrary constant and hence find F_r, the only nonzero component of the force, acting on the particle at time t.

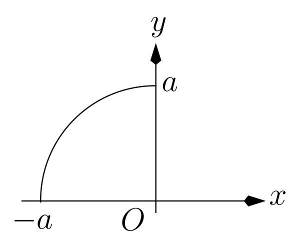
[8 marks]

8. (a) A uniform solid paraboloid of mass M and of height $2a$ is placed symmetrically about the $z-$axis. In terms of cyclindrical polar coordinates: $x = r\cos\phi$, $y = r\sin\phi$, $z = z$, the equation of the curved surface of this paraboloid is given by $z = 2r^2/a$, where $0 \le z \le 2a$, $0 \le \phi \le 2\pi$.
Identify the principal axes of inertia for this paraboloid at the origin O.
Now, show that the volume of the paraboloid is πa^3 and hence find the moment of inertia of the paraboloid about the $z-$axis, i.e. I_{Oz}.



[8 marks]

(b) The Figure shows a rigid uniform lamina of mass M in the $xy-$plane in the shape of a quarter of a circle of radius a in the second quadrant. Ox, Oy are the body axes and the axis Oz is perpendicular to the $xy-$plane, forming a right handed frame.
Obtain the Cartesian equations of the principal axes of inertia at the origin O for the lamina by considering its symmetry properties.



Given that the inertia matrix at O with respect to $Oxyz$ is

$$\frac{Ma^2}{4\pi}\begin{pmatrix} \pi & 2 & 0 \\ 2 & \pi & 0 \\ 0 & 0 & 2\pi \end{pmatrix},$$

write down column matrices representing vectors parallel to the principal axes you found and hence, obtain the principal moment of inertia corresponding to each principal axis.

[7 marks]

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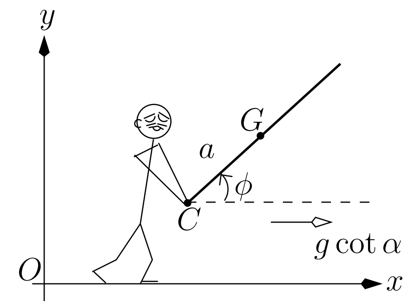
9. A man runs forward, balancing one end C of a long rigid rod of mass M and of uniform thickness in his hands. The end C moves parallel to the horizontal $x-$axis with constant acceleration of magnitude $g\cot\alpha$, $0<\alpha<\pi/2$. The centre of mass G of the rod is at a distance a away from C, and at time t, the rod makes an acute angle ϕ with the horizontal, as shown in the Figure. The motion of the rod is at all times in the $xy-$plane, where $y-$axis is vertically upwards. It is assumed that $Oxyz$ form a right-handed frame.



Given that the rod has a principal axis of inertia through G perpendicular to the $xy-$plane with moment of inertia $Ma^2/3$ about this axis, use the theorem of parallel axis to find its moment of inertia about a parallel axis through C.

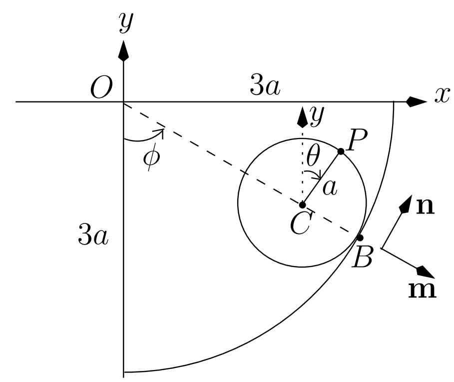
Hence, using the equation for the rotational motion of the rod about the point C, show that

$$4a\sin\alpha\,\ddot{\phi} = 3g\sin(\phi-\alpha).$$

Let $\phi(t)=\alpha+\varepsilon(t)$, where $|\varepsilon|$ is small. Substitute this into the above equation of motion and obtain an approximate solution to the resulting differential equation. Hence verify that the above equilibrium state is dynamically unstable.

[15 marks]

10. The Figure shows a rigid uniform disc of centre C and radius a, rolling inside a fixed disc of centre O and radius $3a$. The motion is planar at all times and two unit vectors $\mathbf{m}$ and $\mathbf{n}$ are introduced along the radial and tangential directions for convenience. At time t, θ is the angle between the upward vertical $y-$axis and CP, where P is a point on the inner circle's rim, and ϕ is the angle between the $y-$axis and OB, where B is the instantaneous point of contact of the two discs, as shown in the Figure.



Find the velocity of the centre point C in terms of a and ϕ, and specify its direction.

Assuming that there is no slipping between the discs, show that $2\dot{\phi} = \dot{\theta}$ and hence apply the law of conservation of energy to further show that

$$3a\dot{\phi}^2 - 2g\cos\phi = \text{const.}$$

[15 marks]

11. At time t, the components of the angular velocity of a uniform rigid body along its principal axes of inertia GX, GY, GZ at its centre of mass G, are $\omega_1(t), \omega_2(t)$ and $\omega_3(t)$ respectively. The corresponding principal moments of inertia at G are $I, 2I$ and I respectively.

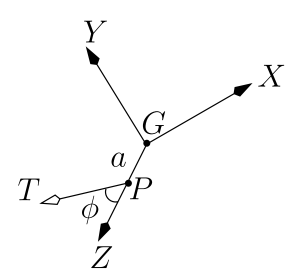
If all the external forces on the body act at G, show that the angular momentum of the body at G is constant.

Hence use the Euler's form to show that the Euler equations for the three-dimensional motion of the body can be written as

$$\dot{\omega}_1 = s\omega_3, \quad \omega_2 = s, \quad \dot{\omega}_3 = -s\omega_1,$$

where s is a constant. Hence, find the differential equations ω_1 and ω_3 satisfy and solve them.

Now, consider the case when there exists a force of constant magnitude T on the $YZ-$plane acting at a point P on GZ, a distance a away from G, making a constant angle ϕ with the GZ axis, as shown in the Figure. Show that the component of the angular velocity ω_3 can now be found as



$$w_3(t) = C_1\cos(st + C_2) + \frac{aT}{sI}\sin\phi, \qquad C_1, C_2 = \text{const.}$$

[15 marks]

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FORMULAE SHEET

Note: In the formulae listed below, C is a body point whereas O is a point fixed in space, and dot denotes the time derivative.

`Centre of mass:` $\mathbf{r}_G = \dfrac{1}{V}\displaystyle\int_V \mathbf{r}\,dV$

`Inertia matrix:` $\dfrac{M}{V}\displaystyle\int_V \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dV$

`Angular velocity:` $\mathbf{v}_P = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{CP}$

`Angular momentum:` $\mathbf{L}_C = I_1\omega_1\mathbf{I} + I_2\omega_2\mathbf{J} + I_3\omega_3\mathbf{K}$

$$\mathbf{L}_O = M(\mathbf{r}_G \times \mathbf{v}_G) + \mathbf{L}_G$$

`Kinetic energy:` $T_C = \dfrac{1}{2}\,(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$

$$T_O = \frac{1}{2}\,M|\mathbf{v}_G|^2 + T_G$$

`Equations of motion:` $\sum_i \mathbf{F}_i = M\dot{\mathbf{v}}_G$

$$\sum_i \mathbf{CP}_i \times \mathbf{F}_i = M(\mathbf{CG} \times \dot{\mathbf{v}}_C) + \dot{\mathbf{L}}_C$$

$$\sum_i \mathbf{OP}_i \times \mathbf{F}_i = \dot{\mathbf{L}}_O$$

`Euler form:` $\dot{\mathbf{L}}_C = [I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3]\,\mathbf{I}$

$$+[I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1]\,\mathbf{J}$$

$$+[I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2]\,\mathbf{K}$$