PAPER CODE NO.	EXAMINER:	
MATH228	DEPARTMENT:	TEL. NO:

THE UNIVERSITY *of* LIVERPOOL

SUMMER 2006 EXAMINATIONS

Bachelor of Science: Year 2
Master of Mathematics: Year 2
Master of Physics: Year 2
No qualification aimed for: Year 1

MECHANICS AND RELATIVITY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. All answers to Section A and the best THREE questions from Section B will be taken into account. Section A carries 55% of the available marks. The marks shown against the sections indicate their relative weights.

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SECTION A

1. A rigid uniform lamina, in the shape of a sector of a disc of mass M, lies in the $xy-$plane. The sector starts at $\theta = \pi/4$ in the first quadrant and sweeps $\pi/2$ with radius a.

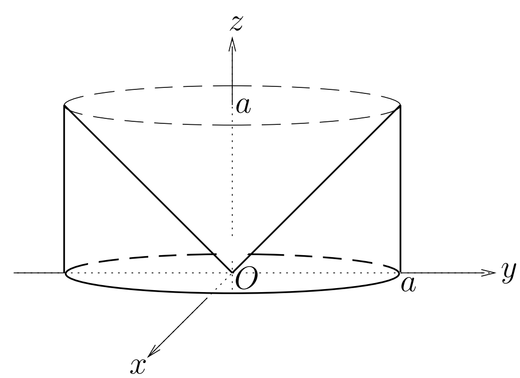
   Draw a clear diagram of the lamina. [1 mark]

   Find, by the use of polar coordinates or otherwise, the coordinates of the centre of mass of the lamina. [6 marks]

   The lamina is now rotated through 2π about the $x-$axis. Show, by the use of the Theorem of Pappus-Guldin or otherwise, that the volume swept out is given by $2\sqrt{2}\pi a^3/3$. [2 marks]

   Hint Centre of mass: $\mathbf{r}_G = \dfrac{1}{A}\displaystyle\int_A \mathbf{r}\, dA$

2. A rigid uniform solid of revolution of mass M is placed symmetrically about the $z-$axis. As shown in the figure, the solid is enclosed by the cylinder, whose curved surface, in terms of cylindrical polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, is given by the equation $r = a$. The solid is bounded above by a cone and below by the $xy-$plane. Again, in terms of cylindrical polar coordinates, the equation of the curved surface of the cone is given by $z = r$.



   Identify the principal axes of inertia for this solid at the origin O, stating briefly your reasons.

   [2 marks]

   Using a multiple integral, show that the volume of the solid is $2\pi a^3/3$.

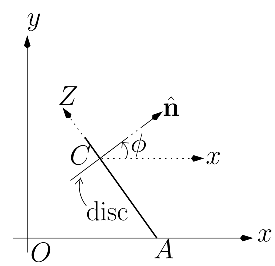
   Hence, deduce that the moment of inertia of the solid about the $z-$axis, i.e. I_{Oz} is $3Ma^2/5$.

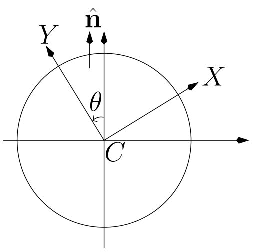
   [7 marks]

   Hint Inertia matrix: $\dfrac{M}{V}\displaystyle\int_V \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dV.$

3. **i.** Define the angular velocity of a rigid uniform body in terms of an arbitrary body vector. [2 marks]

ii. A rigid uniform circular disc is attached to a rigid uniform thin rod at its centre C, about which it rotates, as shown in the top figure. The rod swings freely, about its end A, a fixed point along the $x-$axis, remaining in the $xy-$plane at all times. The choice of the fixed coordinate axes $Oxyz$, forming a right-handed frame, is also shown in the top figure. $\hat{\mathbf{n}}$ is a unit vector in the $xy-$plane and ϕ is the angle between the $x-$axis and the plane of the disc, at time t. CX, CY, CZ are mutually perpendicular body axes, again forming a right-handed frame. A top-on view of the disc is shown in the bottom figure, showing CX, CY lying on the disc. As shown, θ is the angle between $\hat{\mathbf{n}}$ and CY at time t.





Identify the unit vector along the horizontal axis in the bottom figure.

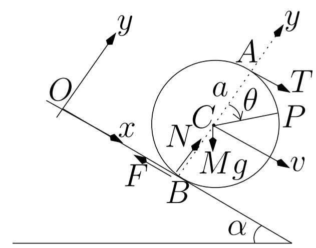
[1 mark]

Hence, show that, at time t the angular velocity of the disc is given by

$$\boldsymbol{\omega} = \dot{\phi}\cos(\theta)\,\mathbf{I} - \dot{\phi}\sin(\theta)\,\mathbf{J} + \dot{\theta}\,\mathbf{K},$$

where $\mathbf{I}, \mathbf{J}$ are unit vectors along CX, CY and $\mathbf{K}$ is a unit vector along the axis of the rod CZ. [6 marks]

4. A rigid uniform disc of radius a and mass M is rolling down <u>without slipping</u> on a plane which is inclined at an angle α to the horizontal, as shown in the figure. B is the instantaneous point of contact between the disc and the inclined plane, and C is the centre of the disc, which moves with constant speed v in the positive $x-$direction. The choice of the coordinate axes, forming a right handed frame, is also shown in the figure. The motion of the disc is restricted to the $xy-$plane at all times. θ is the angle between Cy and $\mathbf{CP}$, were P is a fixed point on the rim of the disc.



In addition to frictional, normal and gravitational forces, a thrust T, parallel to the positive $x-$axis, is applied at point A, as shown.

Show that the angular momentum of the disc about the point B is

$\mathbf{L}_B = -(3Ma^2\dot{\theta}/2)\,\mathbf{k}$. [4 marks]

Hence, using the equation for rotational motion about B and the no-slip condition, deduce that

$$\frac{2}{3}\frac{Mg\,\sin(\alpha) + 2T}{Ma} = \ddot{\theta}.$$ [3 marks]

Finally, show that the displacement of the disc is given by

$$x(t) = \frac{1}{3}\frac{Mg\,\sin(\alpha) + 2T}{Ma}\,t^2,$$

given that the disc starts rolling down from rest from the origin O.

[4 marks]

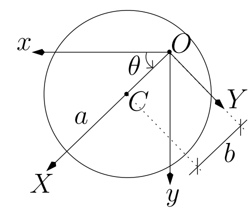
<u>`Hint`</u>

`Theorem of Parallel Axes:` $I_{\parallel} = I_G + Md^2$

`Equations of motion:` $\sum_i \mathbf{CP}_i \times \mathbf{F}_i = M(\mathbf{CG} \times \dot{\mathbf{v}}_C) + \dot{\mathbf{L}}_C$

`Angular velocity:` $\mathbf{v}_P = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{CP}$

5. The figure shows a rigid uniform disc of mass M and radius a, smoothly hinged at a point O, a distance b away from the centre of the disc C. The disc rotates about this point, remaining in the $xy-$plane at all times. Ox is the horizontal axis, Oy is vertically downwards and $Oxyz$ forms a fixed right-handed frame. $OXYZ$, the mutually perpendicular body axes, also form a right-handed frame. At time t, θ is the angle between OX and Ox.



The disc is released at time $t = 0$, from $\theta = 0$ with $\dot{\theta} = \Omega$.

Show, by applying the law of conservation of energy, that

$$\dot{\theta} = \sqrt{\Omega^2 + \frac{2Mgb}{I_{Oz}}\sin(\theta)},$$

where I_{Oz} is the moment of inertia of the disc at O about the $z-$axis.

[9 marks]

`Hint` `Kinetic energy:` $T_0 = \frac{1}{2}M|\mathbf{v}_0|^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}_0$

6. Within the theory of special relativity, the $y-$ and $z-$axes of an inertial frame S are parallel to the corresponding axes of another inertial frame S'. The $x-$axes of S and S' are collinear and S' moves with constant velocity v relative to S.

i. With reference to S and S', state briefly the Principle of Relativity and the fundamental postulate concerning the velocity of light c.

[4 marks]

ii. Two events which take place at different spatial locations in S are observed from S' to occur at the same time t' relative to S'. Show that the events cannot occur simultaneously with respect to S.

[4 marks]

`Hint`

`Lorentz transformation:`

$\Delta x' = \gamma(\Delta x - v\Delta t)$, $\Delta t' = \gamma(\Delta t - v\Delta x/c^2)$ `where` $\gamma = 1/\sqrt{1 - v^2/c^2}$.

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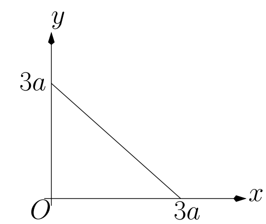
SECTION B

7. The diagram shows a rigid uniform lamina of mass M in the $xy-$plane in the shape of a triangle with sides $3a$ along the coordinate axes, as shown in the figure. Ox, Oy are the body axes and the axis Oz is perpendicular to the $xy-$plane, which form a right handed frame.



Obtain the Cartesian equations of the principal axes of inertia at the origin O for the lamina by considering its symmetry properties.

[2 marks]

Show that the inertia matrix at O with respect to $Oxyz$ is given by

$$\frac{3Ma^2}{4}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

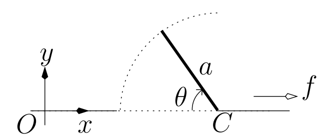
[7 marks]

Write down column matrices representing vectors parallel to the principal axes you found. Hence, obtain the principal moment of inertia corresponding to each principal axis.

[6 marks]

`Hint You may refer to the inertia matrix given in question 2.`

8. The figure shows the vertical cross-section of the top of a railway carriage which has a horizontal roof with a square trapdoor of mass M, side length a and uniform thickness, hinged smoothly to the roof at C and swings freely in the $xy-$plane. The choice of the coordinate axes is also shown in the figure.



Assuming that the carriage is moving with constant acceleration f in the positive $x-$direction, use the equation of motion involving the angular momentum of the trapdoor about the point C to show that:

$$I\ddot{\theta} = -\frac{Maf}{2}\sin(\theta) - \frac{Mga}{2}\cos(\theta),$$

where θ is the angle between the trapdoor and its closed position at time t, and I is its moment of inertia about the horizontal axis through C.

[6 marks]

Given that, at time $t = 0$, the trapdoor was at rest relative to the carriage roof with $\theta = \pi/2$, deduce from the above equation of motion that the trapdoor will close.

[2 marks]

Now, integrate the above equation of motion with respect to time t and hence show that when the trapdoor closes, the magnitude of its angular velocity will be

$$\sqrt{\frac{Ma(f+g)}{I}}.$$

[7 marks]

<u>`Hint`</u>

`Equations of motion:` $\sum_i \mathbf{CP}_i \times \mathbf{F}_i = M(\mathbf{CG} \times \dot{\mathbf{v}}_C) + \dot{\mathbf{L}}_C$

9. At time t, the components of the angular velocity of a uniform rigid body along its principal axes of inertia GX, GY, GZ at its centre of mass G, are $\omega_1(t), \omega_2(t)$ and $\omega_3(t)$ respectively. The corresponding principal moments of inertia at G are $I, 2I$ and αI respectively, where α is a constant parameter.

If all the external forces on the body act at G, show that the angular momentum of the body at G is constant.

[2 marks]

Hence use the Euler's form to verify that the Euler equations for the three-dimensional motion of the body can be written as

$$\dot{\omega}_1 = (2-\alpha)\,\omega_2\omega_3, \quad 2\,\dot{\omega}_2 = (\alpha-1)\,\omega_3\omega_1, \quad \alpha\,\dot{\omega}_3 = -\omega_1\omega_2.$$

[2 marks]

Given that $\alpha = 1$, show, by using the Euler's equations of motion, that

$$\omega_1\dot{\omega}_1 + \omega_3\dot{\omega}_3 = 0.$$

[2 marks]

By integrating this equation with respect to time show that the magnitude of the angular velocity is a constant of the motion.

[3 marks]

Now, assuming that α can take all values $\alpha > 0$, let $\omega_1 = \varepsilon_1(t), \omega_2 = s + \varepsilon_2(t)$ and $\omega_3 = \varepsilon_3(t)$, where s is constant and ε_i, $i = 1, 2, 3$ are small perturbations from equilibrium at time t. Deduce that the equilibrium state is stable for $\alpha < 2$ and unstable for $\alpha > 2$.

[6 marks]

`Hint`

`Equations of motion:` $\sum_i \mathbf{CP}_i \times \mathbf{F}_i = M(\mathbf{CG} \times \dot{\mathbf{v}}_C) + \dot{\mathbf{L}}_C$

`Euler form:`

$$\dot{\mathbf{L}}_C = [I_1\,\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3]\,\mathbf{I} + [I_2\,\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1]\,\mathbf{J} + [I_3\,\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2]\,\mathbf{K}$$

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10. A symmetric spinning top moves under gravity about a stationary fixed pivot O on its axis of symmetry. The centre of mass G of the top lies on this symmetry axis at a distance a from O.

Draw a clearly labelled figure to define the conventional Euler angles θ, ϕ and ψ which specify the position of the top relative to fixed space axes $Oxyz$, where Oz is vertically upwards. What do the time derivatives of Euler angles characterise?

[3 marks]

It may be assumed that the mass and the principal moments of inertia of the top are such that its motion at time t is given by the equations:

$$\begin{aligned} \dot{\psi} + \dot{\phi}\cos(\theta) &= s, \qquad (i) \\ s\cos(\theta) + 3\,\dot{\phi}\sin^2(\theta) &= A, \qquad (ii) \\ \dot{\theta}^2 + \dot{\phi}^2\sin^2(\theta) + 2\,a\cos(\theta) &= B, \qquad (iii) \end{aligned}$$

where $s \neq 0, A$ and B are constants of motion.

The above equations of motion allow existence of a state of dynamical equilibrium in which the top precesses indefinitely, with OG rotating about Oz with constant angular velocity Ω, and is inclined at a constant angle α to Oz, where $0 < \alpha < \pi$. Verify this, explaining briefly.

[2 marks]

Now, by differentiating equations (ii) and (iii) with respect to time, eliminate the terms involving $\ddot{\phi}$, and use equation (i) to show that

$$\ddot{\theta} = [\frac{2}{3}\cos(\theta)\,\dot{\phi}^2 - \frac{1}{3}\dot{\psi}\,\dot{\phi} + a]\sin(\theta).$$

[7 marks]

Finally, find the requirement for the top to precess steadily.

[3 marks]

11. Explain briefly what is meant by an inertial frame of reference.

[3 marks]

Within the context of the theory of special relativity, the inertial frame S' moves parallel to the $x-$axis of the inertial frame S with constant velocity v. The frames S and S' are in “standard configuration”. Explain what this means.

[4 marks]

An observer in S sees a fast moving vehicle travelling along the $x-$axis with constant velocity u and notes that at time $t = 0$ it is at a distance d away from the origin of S. Another observer, moving with S', notes that at time $t' = T$ in S the vehicle is at distance $3d$ from the origin of S'.

Show that

$$T = \frac{d}{u-v}\,[\,3 - 3\frac{uv}{c^2} - \sqrt{1-\frac{v^2}{c^2}}\,].$$

[8 marks]

`Hint You may refer to Lorentz transformation given in question 6.`