SECTION A

1. The utility function U is defined in commodity space by

$$U(x, y) = xy + 2x + y.$$

Express the indifference curve $U(x, y) = U_0$ in the explicit form

$$y = f(x, U_0),$$

for some function f.
Verify that

$$\frac{df}{dx} < 0 \text{ and } \frac{d^2 f}{dx^2} > 0,$$

within the domain of U.
Sketch the indifference curve corresponding to $U_0 = 4$. [6 marks]

2. A utility function W is defined in commodity space by

$$W(x, y) = xy^2 + x^2 y + 3x.$$

A consumer's purchases are subject to the budget constraint

$$x + y \leq 3,$$

where x and y denote the amounts of two commodities.
Determine the amounts purchased. [6 marks]

3. The production function for a single commodity, per unit time, is

$$q(x_1, x_2) = (x_1 - 1)\sqrt{(x_2 + 1)},$$

where x_1 and x_2 are the quantities of the two inputs used.
Sketch a few isoquants for q. Explain why one of the inputs is essential for production, whereas the other is inessential.
Sketch also the production curve for the second input, if $x_1 = 3$. [7 marks]

4. The total cost function for a single-commodity firm is

$$C(q) = 2q^3 - 12q^2 + 30q + 15,$$

where q is the quantity of the commodity produced in unit time.
Determine

(i) The fixed cost;
(ii) The marginal cost function, $MC(q)$;
(iii) The average variable cost function $AVC(q)$.

Find the price at which the firm would be forced to cease production in the *short-run* in an ideal competitive market.

[6 marks]

5. A firm making one good to sell in an ideal competitive market will not supply the market unless the price exceeds a certain value p_0. Above that price the daily supply function is

$$S(p) = \frac{10(2p^2 - 1)}{3 + 2p^2}.$$

where p is the price per unit of the good.
State the value of p_0. If the demand function for the market is

$$D(p) = 3 - p,$$

for $p < 3$, determine the equilibrium price. [6 marks]

6. A monopolist has cost function

$$C(q) = 2q^3 - 12q^2 + 30q + 15,$$

for the production of a quantity q of a good, in unit time. Write down the profit function, as a function of q, when the demand function is

$$D(p) = 38 - p, \quad p < 38.$$

Determine the price charged by the monopolist, and calculate the corresponding profit.
Verify that your answer maximises the monopolist's profit function. [7 marks]

7. The population density, $n(t)$, of a species of fish satisfies

$$\frac{dn}{dt} = 27n - 12n^2 + n^3.$$

Find the equilibrium densities, and determine the stability of each. [5 marks]

8. A mathematical model of the behaviour of two interacting species X and Y is described by the coupled differential equations

$$\frac{dx}{dt} = x(8 - 2x) - 3xy, \quad \frac{dy}{dt} = -y + xy,$$

where $x(t)$ and $y(t)$ are the population densities of X and Y, respectively.
Explain the biological significance of the terms which appear in these equations.
Find the equilibria - you are not required to classify them. [6 marks]

9. The community matrix at a particular equilibrium point of a two-species model has real eigenvalues of opposite sign. Explain briefly, with the aid of a diagram, why the equilibrium is a saddle point. [6 marks]

SECTION B

10. In a unit of time, a consumer's desire for the quantities x and y of swedes and turnips, respectively, is given by the utility function

$$U(x,y) = \frac{xy}{2x+y}.$$

Sketch the indifference curve $U = 1$. For which commodity bundle, on this indifference curve, does the rate of commodity substitution equal 1 ?
Unit quantities of swedes and turnips cost p_1 and p_2 units of money, respectively. In a unit of time the consumer has B units of money to spend on these root vegetables. Show that the demand function for swedes is

$$\frac{B}{p_1 + \sqrt{2p_1p_2}}.$$

If the consumer has a fixed budget, and the price of turnips is also constant, determine whether the expenditure on swedes increases or decreases in response to an increase in p_1. [15 marks]

11. An ideal competitive market for motor boats is supplied by 10 firms. Each firm has a maximum manufacturing capacity of 35 boats per month. Below that production level, each firm has monthly supply function

$$S(p) = \begin{cases} 7p - 14, & \text{if } p \geq 2 \\ 0, & \text{if } p < 2, \end{cases}$$

where p kilopounds is the price of a motor boat.
Sketch the supply function for each firm.
The aggregate monthly demand function for motor boats is

$$D(p) = \begin{cases} 10(30 - 4p), & \text{if } p \leq 7.5 \\ 0, & \text{if } p > 7.5. \end{cases}$$

Determine the market price for motor boats, and the monthly production.
Motor boats become so popular that the government decides to impose a tax at the fractional rate t of the market price. Determine the new market price if $t = 0.25$.
Because of an election pledge to the *safety at sea* lobby, a new government decides to kill the market in motor boats. What tax rate would achieve this aim?

[15 marks]

12. Two companies make the same good. Their cost functions are :

$$C_1(q_1) = q_1^2 + 3q_1 + 2,$$

$$C_2(q_2) = q_2^2 + 4q_2 + 4,$$

where q_1 and q_2 represent the quantities of the good produced by the two companies per hour.

The two companies, acting as a Cournot duopoly in a market with demand function $D(p) = 12 - p$, try to maximise their profits independently of each other. Show that the optimal outputs are approximately $q_1 = 1.87$ and $q_2 = 1.53$ units per hour.

Find the price charged per unit of the good and the profits made by the companies.

The firms agree to cooperate and maximise their joint profit. Find their new outputs, their new individual profits and the new price.

Comment very briefly on your results. [15 marks]

13. A population of pernicious animals would, if left unchecked, increase in accordance with a Malthusian growth law, with relaxation time τ.
The population of animals is denoted by $n(t)$, at time t. The initial population is n_0.
A group of bounty hunters proposes two strategies to keep the population in check :
Strategy 1 : Hunt at the rate of $\frac{N}{\tau}$ individuals per unit time, where N is a positive constant;
Strategy 2 : At each of the times $\tau, 2\tau, 3\tau \cdots$, remove exactly N individuals from the population.
The law forbids that any population of animals, however malevolent, should ever be hunted to extinction. Assuming that this law is strictly adhered to by the bounty hunters, compare the resulting populations at time $t = \tau$ under each of the two hunting strategies. In particular, state which is the greater?
Whilst the law abiding bounty hunters are arguing about their strategy, a rogue group of hunters steps in and, adopting strategy 2, causes the animal population to become extinct after exactly N individuals have been removed at time $t = 3\tau$. If the initial population n_0 is 100 animals, determine N.

[15 marks]

14. Two organisms in a certain relationship change with time according to the coupled differential equations :

$$\frac{dx}{dt} = x(12 - 4x + y), \quad \frac{dy}{dt} = y(12 - 6y + 3x),$$

where $x(t)$ and $y(t)$ represent the respective population densities of the two organisms.

Explain the biological significance of the terms which appear on the right hand sides of these equations.

Show that the system has equilibria at $(0,0), (0,2), (3,0)$, and $(4,4)$.

Show also that the only stable equilibrium is at $(4,4)$.

Find the zero and infinite isoclines, and show that there is also an isocline of the form $y = kx$, for some $k > 0$.

Draw a phase plane diagram showing the equilibria, and the three isoclines.

Sketch trajectories on your diagram which describe the behaviour of the system. [15 marks]