MATH227 MATHEMATICAL METHODS FOR NON-PHYSICAL SYSTEMS

JANUARY 2007

Candidates should attempt the whole of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

SECTION A

1. The utility function U is defined by

$$U(x, y) = xy + 7x + 3y, \quad x \geq 0, \quad y \geq 0.$$

Express the indifference curve $U(x, y) = U_0$ in the explicit form

$$y = f(x, U_0),$$

for some function f.

Verify that

$$\frac{df}{dx} < 0 \quad \text{and} \quad \frac{d^2 f}{dx^2} > 0,$$

within the domain of U.

[4 marks]

2. A consumer has utility function

$$U(x, y) = (x + 3)^3 (y + 2)^5, \quad x \geq 0, \quad y \geq 0,$$

where x and y denote the amounts of two commodities. Determine the amounts which would be purchased in order to maximise $U(x, y)$ subject to the budget constraint

$$2x + 3y = 36.$$

[5 marks]

3. A firm has production function

$$q(x, y) = \frac{(x + 5)(y + 4)}{x + y + 9}, \quad x \geq 0, \quad y \geq 0,$$

and cost function

$$c(x, y) = 16x + 9y + 4$$

per unit time, with inputs x and y. Find the minimum value of $c(x, y)$ consistent with a fixed production level of 3 units per unit time.

[7 marks]

4. The total cost function for a single-commodity firm is

$$C(q) = q^3 - 6q^2 + 14q + 10,$$

where q is the quantity of the commodity produced in unit time.

Determine:

(i) The fixed cost;

(ii) The marginal cost function, $MC(q)$;

(iii) The average variable cost function, $AVC(q)$.

Find the price at which the firm would be forced to cease production in the *short-run* in an ideal competitive market.

[6 marks]

5. In an ideal competitive market for one good, the weekly supply function is given by

$$S(p) = 4\frac{5p + 2}{5p + 8}$$

where p is the price per unit of the good, and the demand function is

$$D(p) = \sqrt{21 - 3p},$$

for $p < 7$, where p is the price per unit good.

Verify that $S(p)$ is an increasing function of p.

The tax rate is set at 20%. Determine the equilibrium price.

[7 marks]

6. A monopolist has cost function

$$C(q) = q^3 - 8q^2 + 24q + 3,$$

for the production of a quantity q of a commodity, per unit time. Write down the profit function, as a function of q, when the demand function is

$$D(p) = 48 - p, \quad p < 48.$$

Determine the market price. Verify that your answer maximises the monopolist's profit function for $q \geq 0$.

[8 marks]

7. Suppose that a worker has an amount of leisure time x (as a fraction of a day) and an income of y (in £ per day). Her rate of pay is p (in £ per day). Her utility function is given by

$$U(x, y) = xy + 9x + 2y.$$

If x and y are chosen to maximise U, determine x for $p < 9$ and show that it decreases with p.

[5 marks]

8. The population density, $n(t)$, of a species of fish satisfies

$$\frac{dn}{dt} = -21n + 10n^2 - n^3.$$

Find the equilibrium densities, and determine their stability.

[4 marks]

9. A mathematical model of the behaviour of two interacting species X and Y is described by the coupled differential equations

$$\frac{dx}{dt} = x(4 - 2x + 3y), \quad \frac{dy}{dt} = y(3 + x - 4y),$$

where $x(t)$ and $y(t)$ are the population densities of X and Y respectively. Find the equilibria–you are not required to classify them.

[5 marks]

10. The community matrix at a particular equilibrium point of a two-species model has complex eigenvalues with negative real parts. Describe briefly, with the aid of diagrams, the possible local behaviour of trajectories near to the equilibrium point.

[4 marks]

SECTION B

11. Sam's preferences for pies and sandwiches are quantified by the utility function

$$U(x, y) = (x + 9)^{\frac{1}{3}}(y + 5)^{\frac{2}{3}}, \quad x \geq 0, \quad y \geq 0,$$

where x and y denote the numbers of pies and sandwiches eaten per week, respectively.

Pies cost £p each and sandwiches cost £q each. Sam has £15 per week to spend on these foods. Show that

$$x(p, q) = \frac{15 + 5q}{3p} - 6, \quad \text{for} \quad 18p - 5q < 15.$$

Find a corresponding expression for $y(p, q)$ and state where it is valid.

The own-elasticity of Sam's demand for pies, ϵ_x, is defined by

$$\epsilon_x = \frac{p}{x}\frac{dx}{dp},$$

where q is regarded as a constant. Compute ϵ_x and show that $\epsilon_x < -1$ if $18p - 5q < 15$.

Sketch the demand curve $x(p, 0)$.

[15 marks]

12. Each of N identical firms producing a single commodity has cost function

$$C(q) = q^3 - 2q^2 + 5q + 36.$$

Show that each firm has supply function

$$S(p) = \begin{cases} \frac{2+\sqrt{3p-11}}{3} & \text{if } p \geq 4 \\ 0 & \text{if } p < 4. \end{cases}$$

Find the equilibrium price and the profit/loss made by each firm in an ideal competitive market where the demand function is

$$D(p) = N(11 - p).$$

Find the price below which production is not viable in the *long-run*.

[15 marks]

13. Two firms producing the same commodity have cost functions

$$C_1(q_1) = 7 + 6q_1 + q_1^2,$$
$$C_2(q_2) = 5 + 9q_2 + q_2^2,$$

where q_1 and q_2 are the respective quantities of production per unit time.

The two firms form a Cournot duopoly to supply a market which has demand function $D(p) = 15 - p$. Find the production of each firm. Find the price charged per unit of the good and the profits made by the companies.

The firms tentatively agree to co-operate and maximise their joint profit. Find what the individual profits would be, if such an agreement were carried out.

[15 marks]

14. A harvesting experiment is performed on two groups of the same species of fish. In both cases it is assumed that the population density would increase exponentially, with relaxation time T, if left alone. The initial population density of each group is N_0.

The first population is harvested at a constant rate, so that the density $n_1(t)$ changes according to

$$\frac{dn_1}{dt} = \frac{1}{T}(n_1 - N),$$

where N is a constant and $0 < N < N_0$.

The second population density, $n_2(t)$, is allowed to change according to

$$\frac{dn_2}{dt} = \frac{n_2}{T},$$

but at times T, $2T$, $3T, \ldots$ the population is harvested in such a way that the population density is multiplied by a factor of $\frac{1}{2}$.

For each harvesting strategy, find the population density immediately after the times T, $2T$, $3T, \ldots jT$, j an integer.

Deduce that both populations will grow indefinitely; but that the second population will die out if the population density is multiplied by $\frac{1}{3}$ rather than $\frac{1}{2}$ after each interval T.

[15 marks]

15. The population densities of two interacting species evolve with time according to

$$\frac{dx}{dt} = x(7-2x) - 3xy, \quad \frac{dy}{dt} = y(10-2y) - 4xy,$$

where $x(t)$ and $y(t)$ are the population densities of the two species.

Comment on the biological significance of each of the terms on the right hand sides of these equations.

Find the equilibria and classify all except the coexistence equilibrium.

Let the coexistence equilibrium be (x_c, y_c). Write down the linearised growth equations in the neighbourhood of (x_c, y_c), using the substitutions $x = x_c + \epsilon_x$, $y = y_c + \epsilon_y$ where the deviations from equilibrium are small. Verify that the particular solutions corresponding to the initial conditions $x(0) = x_c + \delta$, $y(0) = y_c + 4\delta$ are given approximately by

$$x(t) = x_c + \delta\left[3e^{-8t} - 2e^{2t}\right], \quad y(t) = y_c + 2\delta\left[e^{-8t} + e^{2t}\right].$$

[15 marks]