MATH227 MATHEMATICAL METHODS FOR NON-PHYSICAL SYSTEMS

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**1.**

$$U(x,y) = xy + 5x + 4y = U_0 \Rightarrow y = \frac{U_0 - 5x}{x+4} = -5 + \frac{U_0 + 20}{x+4}.$$

$$\frac{dy}{dx} = -\frac{U_0+20}{(x+4)^2} < 0,$$

$$\frac{d^2y}{dx^2} = 2\frac{U_0+20}{(x+4)^3} > 0,$$

**2.** Budget constraint touches indifference curve where

$$\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} = \frac{3}{2}$$

$$\frac{4(x+5)^3(y+7)^3}{3(x+5)^4(y+7)^2} = \frac{3}{2}$$

$$8(y+7) = 9(x+5) \Rightarrow 9x - 8y = 11.$$

Solving with $3x + 2y = 13$, we have $x = 3$, $y = 2$. Then $U = U_0 = (3+5)^4(2+7)^3 = 8^4.9^3$.

**3.** $c(x,y)$ is minimised where

$$\frac{\frac{\partial q}{\partial x}}{\frac{\partial q}{\partial y}} = \frac{\frac{\partial c}{\partial x}}{\frac{\partial c}{\partial y}}$$

$$\frac{\frac{1}{3}(x+1)^{-\frac{2}{3}}(y+2)^{\frac{2}{3}}}{\frac{2}{3}(x+1)^{\frac{1}{3}}(y+2)^{-\frac{1}{3}}} = \frac{27}{2}$$

$$\frac{1}{2}\frac{y+2}{(x+1)} = \frac{27}{2} \Rightarrow y+2 = 27(x+1)$$

$$\Rightarrow 27^{\frac{2}{3}}(x+1) = 12 \Rightarrow 9(x+1) = 12 \Rightarrow x = \tfrac{1}{3}, \quad y = 34$$

$$\Rightarrow c = 27x + 2y + 4 = 9 + 68 + 4 = 81.$$

Minimum cost for production level of 12 units is 81 units.

**4.**

$$C(q) = q^3 - 8q^2 + 24q + 12.$$

(i) Fixed cost $C(0) = 12$. (ii) $MC(q) = C'(q) = 3q^2 - 16q + 24$.

(iii) $AVC(q) = \frac{C(q)-C(0)}{q} = q^2 - 8q + 24$.

Cease production when $p = \min(AVC)$.

$$AVC'(q) = 2q - 8 = 0 \quad \text{when} \quad q = 4 \Rightarrow p = AVC(4) = 8.$$

**5.**

$$S(p) = 18\frac{2p+1}{4p+3} \Rightarrow \frac{dS}{dp} = 18\frac{2(4p+3) - 4(2p+1)}{(4p+3)^2} = \frac{36}{(4p+3)^2} > 0.$$

For a tax-rate of $t$, equilibrium is where

$$S((1-t)p) = D(p) \Rightarrow 18\frac{2.\frac{3}{4}p + 1}{4.\frac{3}{4}p + 3} = 2\sqrt{20 - 2p}$$

$$\Rightarrow 3\frac{3p+2}{p+1} = 2\sqrt{20 - 2p}.$$

$p = 2$ is a solution by inspection. Since $S(p)$ is increasing, $D(p)$ is decreasing, it is the only solution. Then amount sold in a week $= D(2) = 8$.

**6.**

$$C(q) = q^3 - 6q^2 + 14q + 3, \quad D(p) = 22 - p = q \Rightarrow p = 22 - q.$$

Profit given by

$$P(q) = pq - C(q) = (22 - q)q - (q^3 - 6q^2 + 14q + 3) = -\, q^3 + 5q^2 + 8q - 3$$

$$\Rightarrow P'(q) = -3q^2 + 10q + 8 = -(3q + 2)(q - 4) = 0 \quad \text{for} \quad q = -\tfrac{2}{3}, \quad 4.$$

Take the +ve solution $q = 4$. Then $p = 22 - q = 18$.

$$P''(q) = -6q + 10 < 0 \quad \text{for} \quad q = 4.$$

So we have a local maximum. Also

$$P(4) = -4^3 + 5.4^2 + 8.4 - 3 = 45 > P(0) = -3.$$

So $q = 4$ is a global maximum.

**7.**

$$\frac{dn}{dt} = -20n + 9n^2 - n^3 = -n(n-4)(n-5) = f(n).$$

Equilibrium densities $n = 0$, $n = 4$, $n = 5$.

$f'(0) < 0$, $f(n) \to -\infty$ as $n \to \infty$. So graph looks like this:

Equilibria at $n = 0$, $n = 5$ stable; equilibrium at $n = 4$ unstable.

**8.**

$$\frac{dx}{dt} = x(1 - 5x + 3y), \quad \frac{dy}{dt} = y(6 + 3x - 4y),$$

$$\begin{aligned} 1 - 5x + 3y = 6 + 3x - 4y =&0 \Rightarrow x = 2, \quad y = 3, \\ \text{or} \quad y = 1 - 5x + 3y =&0 \Rightarrow x = \tfrac{1}{5}, \\ \text{or} \quad x = 6 + 3x - 4y =&0 \Rightarrow y = \tfrac{3}{2}, \\ \text{or} \quad x = y =&0. \end{aligned}$$

So the equilibria are $(2,3)$, $(\frac{1}{5},0)$, $(0,\frac{3}{2})$, $(0,0)$.

**9.**

$$\mathbf{x} = \mathbf{x}^e + c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t},$$

where $\lambda_{1,2}$ are the e-values, $\mathbf{x}_{1,2}$ are the e-vectors. If $\lambda_1$ and $\lambda_2$ have the same signs, then we have an improper node; stable if the e-values are $-$ve, unstable if $+$ve.

**10.**

$$U(0,N) = 5N + 2 > U(N,0) = 3N + 2$$

so $N$ cups of coffee preferred to $N$ cups of tea.

Indifference curves given by

$$y = \frac{U_0 - 3x - 2}{x + 5}.$$

Budget constraint touches indifference curve where

$$\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} = \frac{p}{q}$$

$$\frac{y+3}{x+5} = \frac{p}{q} \Rightarrow y = \frac{p}{q}(x+5) - 3.$$

Substituting into budget constraint $px + qy = 12$ we have

$$2px + 5p - 3q = 12 \Rightarrow x = \frac{12 - 5p + 3q}{2p} = \frac{12 + 3q}{2p} - \frac{5}{2}$$

$$\Rightarrow y = \frac{12 + 5p - 3q}{2q} = \frac{12 + 5p}{2q} - \frac{3}{2}.$$

$$\epsilon_x = p\frac{2p}{12 - 5p + 3q}\left(-\frac{12 + 3q}{2p^2}\right) = \frac{12 + 3q}{5p - 12 - 3q}$$

$$\Rightarrow \epsilon_x + 1 = \frac{5p}{5p - 12 - 3q} < 0 \quad \text{if} \quad 5p - 3q < 12.$$

**11.**

$$C(q) = q^3 - 4q^2 + 6q + 64 \Rightarrow AVC(q) = q^2 - 4q + 6.$$
$$AVC'(q) = 2q - 4 = 0 \quad \text{when} \quad q = 2 \Rightarrow \min(AVC) = 2.$$

So cease production when $p = \min(AVC) = 2$. For $p \geq 2$,

$$p = C'(q) = 3q^2 - 8q + 6 \Rightarrow 3q^2 - 8q + 6 - p = 0$$
$$\Rightarrow q = \frac{8 \pm \sqrt{64 - 12(6-p)}}{6} = \frac{4 \pm \sqrt{3p-2}}{3}.$$

Take +ve sign for maximum profit. So

$$S(p) = \begin{cases} \frac{4+\sqrt{3p-2}}{3} & \text{if } p \geq 2 \\ 0 & \text{if } p < 2. \end{cases}$$

Equilibrium is when $NS(p) = D(p)$ ($N$ firms) so

$$\frac{4 + \sqrt{3p-2}}{3} = 3 - \tfrac{1}{2}p.$$

$p = 2$ is a solution by inspection, and since $D(p)$ is decreasing and $S(p)$ is increasing, it is unique.

$$p = 2 \Rightarrow q = \frac{1}{N}D(p) = 2 \Rightarrow P(q) = pq - C(q)$$
$$= 4 - (8 - 16 + 12 + 64) = -64.$$

So each firm makes a loss of 64 units.

Production not viable in the long-run for $p < \min(ATC)$.

$$ATC = q^2 - 4q + 6 + \frac{64}{q} \Rightarrow ATC'(q) = 2q - 4 - \frac{64}{q^2}$$
$$= 0 \quad \text{when} \quad q = 4,$$

by inspection. It is a minimum, since

$$ATC''(q) = 2 + \frac{128}{q^3} > 0.$$
$$\min(ATC) = 16 - 16 + 6 + 16 = 22.$$

So minimum price in the long-run is 22 units.

**12.**

$$\begin{aligned} C_1(q_1) =& 5 + 4q_1 + q_1^2, \\ C_2(q_2) =& 7 + 9q_2 + \tfrac{1}{2}q_2^2, \end{aligned}$$

Profits:

$$\begin{aligned} P_1(q_1, q_2) =& pq_1 - (5 + 4q_1 + q_1^2) = [18 - (q_1 + q_2)]q_1 - (5 + 4q_1 + q_1^2) \\ =& -2q_1^2 - q_1q_2 + 14q_1 - 5, \\ P_2(q_1, q_2) =& pq_2 - (7 + 9q_2 + \tfrac{1}{2}q_2^2) = [18 - (q_1 + q_2)]q_2 - (7 + 9q_2 + \tfrac{1}{2}q_2^2) \\ =& -q_1q_2 - \tfrac{3}{2}q_2^2 + 9q_2 - 7. \end{aligned}$$

Cournot duopoly $\Rightarrow$ maximise $P_1$, $P_2$ wrto $q_1$, $q_2$ respectively. So

$$\begin{aligned} \frac{\partial P_1}{\partial q_1} =& -4q_1 - q_2 + 14 = 0, \\ \frac{\partial P_2}{\partial q_2} =& -q_1 - 3q_2 + 9 = 0, \end{aligned}$$

Then $q_1 = 3$, $q_2 = 2$. So $p = 18 - 3 - 2 = 13$ and $P_1(3, 2) = 13$, $P_2(3, 2) = -1$.

If co-operate, maximise

$$\begin{aligned} P(q_1, q_2) =& P_1(q_1, q_2) + P_2(q_1, q_2) \\ =& -\frac{3}{2}q_1^2 - 2q_1q_2 + 14q_1 - 2q_2^2 + 9q_2 - 12 \\ \frac{\partial P_1}{\partial q_1} =& -4q_1 - 2q_2 + 14 = 0, \\ \frac{\partial P_2}{\partial q_2} =& -2q_1 - 3q_2 + 9 = 0, \end{aligned}$$

giving $q_1 = 3$, $q_2 = 1$. Then $P_1(3, 1) = 16$, $P_2(3, 1) = -\frac{5}{2}$.

**13.**

$$\frac{dn}{dt} = 18n^2 - 3n^3 = -3n^2(n-6) = f(n).$$

The equilibrium densities are $n = 0$ and $n = 6$. The graph of $f(n)$ looks like this:

so $n = 0$ is unstable, $n = 6$ stable.

Now

$$\frac{dn}{dt} = 18n^2 - 3n^3 - cn = -n(3n^2 - 18n + c) = g(n).$$

$g(n) = 0$ when

$$n = \frac{18 \pm \sqrt{324 - 12c}}{6} = n_{\pm}, \quad \text{say.}$$

2 real roots provided $c \leq 27$. $g'(n) < 0$ at $n = 0$ implies graph looks like this:

So

$$n_s = n_+ = \frac{18 + \sqrt{324 - 12c}}{6}$$

for stable equilibrium.

For $c > 27$ have no real roots, so graph looks like this:

and $n = 0$ is the only stable equilibrium; the gazelles die out.

Catch

$$C = cn_s = c\frac{18 + \sqrt{324 - 12c}}{6}.$$

$$\frac{dC}{dc} = \frac{18 + \sqrt{324 - 12c}}{6} + \tfrac{1}{2}(-12)\frac{(324 - 12c)^{-\frac{1}{2}}}{6} = 0.$$

$$\Rightarrow \frac{18 + \sqrt{324 - 12c}}{6} = \frac{1}{\sqrt{324 - 12c}}c.$$

satisfied by $c = 24$.

$$\begin{aligned}\frac{d^2C}{dc^2} =& \tfrac{1}{2}(-12)\frac{(324 - 12c)^{-\frac{1}{2}}}{6} + \tfrac{1}{2}(-12)\frac{(324 - 12c)^{-\frac{1}{2}}}{6} + \tfrac{1}{2} \\ &+ \tfrac{1}{2}(-12)\left(-\tfrac{1}{2}\right)(-12)\frac{(324 - 12c)^{-\frac{3}{2}}}{6}c < 0\end{aligned}$$

So $c = 24$ is a maximum.

**14.**

$$\frac{dx}{dt} = x(2-4x) + xy, \quad \frac{dy}{dt} = y(7-3y) - xy,$$

Terms (1), (3) are logistic growth functions, implying each population could survive on its own in a limited resource environment.

Terms (2), (4) with opposite signs imply the first species is preying on the second.

$$\begin{aligned} \text{Either} \quad 2-4x+y &= 7-x-3y = 0 \Rightarrow x = 1, \quad y = 2 \\ \text{or} \quad x = 7-x-3y &= 0 \Rightarrow y = \tfrac{7}{3}, \\ \text{or} \quad y = 2-4x+y &= 0 \Rightarrow x = \tfrac{1}{2}, \\ \text{or} \quad x = y = 0. \end{aligned}$$

So the equilibria are $(0,0)$, $(0,\frac{7}{3})$, $(\frac{1}{2},0)$, $(1,2)$.

Community matrix

$$A = \begin{pmatrix} (2-4x+y)-4x & x \\ -y & (7-x-3y)-3y \end{pmatrix}.$$

For $(0,0)$, $A = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$. E-values 2, 7 both positive $\Rightarrow$ improper node, unstable.

For $(0,\frac{7}{3})$, $A = \begin{pmatrix} \frac{13}{3} & 0 \\ -\frac{7}{3} & -7 \end{pmatrix}$. E-values $\frac{13}{3}$, $-7$ opposite signs $\Rightarrow$ saddle point.

For $(\frac{1}{2},0)$, $A = \begin{pmatrix} -2 & \frac{1}{2} \\ 0 & \frac{13}{2} \end{pmatrix}$. E-values $-2$, $\frac{13}{2}$ opposite signs $\Rightarrow$ saddle point.

For $(1,2)$, $A = \begin{pmatrix} -4 & 1 \\ -2 & -6 \end{pmatrix}$.

Linearised equations

$$\begin{aligned} \frac{d\epsilon_x}{dt} &= -4\epsilon_x + \epsilon_y \\ \frac{d\epsilon_y}{dt} &= -2\epsilon_x - 6\epsilon_y. \\ -4\epsilon_x + \epsilon_y &= e^{-5t}\left[-4\delta\cos t - \delta(\cos t + \sin t)\right] \\ &= e^{-5t}\left[-5\delta\cos t - \delta\sin t\right] = \frac{d\epsilon_x}{dt} \\ -2\epsilon_x - 6\epsilon_y &= e^{-5t}\left[-2\delta\cos t + 6\delta(\cos t + \sin t)\right] \\ &= e^{-5t}\left[4\delta\cos t + 6\delta\sin t\right] = \frac{d\epsilon_y}{dt} \end{aligned}$$

Also $\epsilon_x(0) = \delta$, $\epsilon_y(0) = -\delta$.