MATH227 MATHEMATICAL METHODS FOR NON-PHYSICAL SYSTEMS

JANUARY 2006

Candidates should attempt the whole of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

SECTION A

1. The utility function U is defined by

$$U(x, y) = xy + 5x + 4y, \quad x \geq 0, y \geq 0.$$

Express the indifference curve $U(x, y) = U_0$ in the explicit form

$$y = f(x, U_0),$$

for some function f.

Verify that

$$\frac{df}{dx} < 0 \quad \text{and} \quad \frac{d^2 f}{dx^2} > 0,$$

within the domain of U.

Sketch the indifference curve $U_0 = 20$.

[6 marks]

2. A consumer has utility function

$$U(x, y) = (x + 5)^4 (y + 7)^3$$

and is subject to the budget constraint

$$3x + 2y = 13$$

where x and y denote the amounts of two commodities. Determine the amounts purchased.

Show that the budget constraint line touches the indifference curve $U = 8^4.9^3$.

[6 marks]

3. A firm has production function

$$q(x, y) = (x + 1)^{\frac{1}{3}} (y + 2)^{\frac{2}{3}}$$

and cost function

$$c(x, y) = 27x + 2y + 4$$

per unit time, with inputs x and y. Find the minimum value of $c(x, y)$ consistent with a fixed production level of 12 units per unit time.

[6 marks]

4. The total cost function for a single-commodity firm is

$$C(q) = q^3 - 8q^2 + 24q + 12,$$

where q is the quantity of the commodity produced in unit time.

Determine:

(i) The fixed cost;

(ii) The marginal cost function, $MC(q)$;

(iii) The average variable cost function, $AVC(q)$.

Find the price at which the firm would be forced to cease production in the *short-run* in an ideal competitive market.

[6 marks]

5. In an ideal competitive market for one good, the weekly supply function is given by

$$S(p) = 18\frac{2p+1}{4p+3}$$

where p is the price per unit of the good, and the demand function is

$$D(p) = 2\sqrt{20-2p},$$

for $p < 10$, where p is the price per unit good.

Verify that $S(p)$ is an increasing function of p.

The tax rate is set at 25%. Determine the equilibrium price, and the amount sold in a week.

[8 marks]

6. A monopolist has cost function

$$C(q) = q^3 - 6q^2 + 14q + 3,$$

for the production of a quantity q of a commodity, per unit time. Write down the profit function, as a function of q, when the demand function is

$$D(p) = 22 - p, \quad p < 22.$$

Determine the market price. Verify that your answer maximises the monopolist's profit function for $q \geq 0$.

[8 marks]

7. The population density, $n(t)$, of a species of fish satisfies

$$\frac{dn}{dt} = -20n + 9n^2 - n^3.$$

Find the equilibrium densities, and determine their stability.

[5 marks]

8. A mathematical model of the behaviour of two interacting species X and Y is described by the coupled differential equations

$$\frac{dx}{dt} = x(1 - 5x + 3y), \quad \frac{dy}{dt} = y(6 + 3x - 4y),$$

where $x(t)$ and $y(t)$ are the population densities of X and Y respectively. Find the equilibria–you are not required to classify them.

[5 marks]

9. The community matrix at a particular equilibrium point of a two-species model has distinct negative eigenvalues. Describe briefly, with the aid of diagrams, the possible local behaviour of trajectories near to the equilibrium point.

[5 marks]

SECTION B

10. Pat's preferences for tea and coffee are quantified by the utility function

$$U(x, y) = xy + 3x + 5y + 2,$$

where x and y denote the numbers of cups of tea and cups of coffee drunk per week, respectively.

Show that Pat prefers N cups of coffee, with no tea, to N cups of tea, with no coffee.

Tea costs £p per cup and coffee costs £q per cup. Pat has £12 per week to spend on these drinks. Show that

$$x(p, q) = \frac{12 + 3q}{2p} - \frac{5}{2}.$$

Find a corresponding expression for $y(p, q)$.

The own-elasticity of Pat's demand for tea, ϵ_x, is defined by

$$\epsilon_x = \frac{p}{x}\frac{dx}{dp},$$

where q is regarded as a constant. Compute ϵ_x and show that $\epsilon_x < -1$ if $5p - 3q < 12$.

Sketch the demand curve $x(p, 1)$.

[15 marks]

11. Each of N identical firms producing a single commodity has cost function

$$C(q) = q^3 - 4q^2 + 6q + 64.$$

Show that each firm has supply function

$$S(p) = \begin{cases} \frac{4+\sqrt{3p-2}}{3} & \text{if } p \geq 2 \\ 0 & \text{if } p < 2. \end{cases}$$

Find the equilibrium price and the profit/loss made by each firm in an ideal competitive market where the demand function is

$$D(p) = N(3 - \frac{1}{2}p).$$

Find the price below which production is not viable in the *long-run*.

[15 marks]

12. Two firms producing the same commodity have cost functions

$$C_1(q_1) = 5 + 4q_1 + q_1^2,$$
$$C_2(q_2) = 7 + 9q_2 + \tfrac{1}{2}q_2^2,$$

where q_1 and q_2 are the respective quantities of production per unit time.

The two firms form a Cournot duopoly to supply a market which has demand function $D(p) = 18 - p$. Find the production of each firm. Find the price charged per unit of the good and the profits made by the companies.

The firms tentatively agree to co-operate and maximise their joint profit. Find what the individual profits would be, if such an agreement were carried out.

[15 marks]

13. The population density of gazelles, n, in a well-defined region of a wildlife reserve satisfies

$$\frac{dn}{dt} = 18n^2 - 3n^3.$$

Find the non-zero equilibrium density, and determine its stability.

A pride of lions starts preying upon the gazelles. The growth law for the population density of gazelles becomes modified:

$$\frac{dn}{dt} = 18n^2 - 3n^3 - cn,$$

where c is the number of lions hunting gazelles each day. Given that $c \leq 27$, find the new equilibrium population densities. Identify the non-zero density n_s at which the population is in stable equilibrium.

Explain what would happen if more than 27 lions hunted gazelles each day.

Assuming $c \leq 27$, and the number of gazelles caught each day by each lion is proportional to n_s, find the the value of c which maximises the total number caught daily by the pride.

[15 marks]

14. The population densities of two interacting species evolve with time according to

$$\frac{dx}{dt} = x(2 - 4x) + xy, \quad \frac{dy}{dt} = y(7 - 3y) - xy,$$

where $x(t)$ and $y(t)$ are the population densities of the two species.

Comment on the biological significance of each of the terms on the right hand sides of these equations.

Find the equilibria and classify all except the coexistence equilibrium.

Let the coexistence equilibrium be (x_c, y_c). Write down the linearised growth equations in the neighbourhood of (x_c, y_c), using the substitutions $x = x_c + \epsilon_x$, $y = y_c + \epsilon_y$ where the deviations from equilibrium are small. Verify that the particular solutions corresponding to the initial conditions $x(0) = x_c + \delta$, $y(0) = y_c - \delta$ are given approximately by

$$x(t) = x_c + \delta e^{-5t} \cos t, \quad y(t) = y_c - \delta e^{-5t} [\cos t + \sin t] .$$

[15 marks]