SECTION A

1. The utility function U is defined by

$$U(x, y) = \sqrt{xy}, \quad x \geq 0, \; y \geq 0.$$

Express the indifference curve $U(x, y) = U_0$ in the explicit form

$$y = f(x, U_0),$$

for some function f.
Verify that

$$\frac{df}{dx} < 0 \;\text{ and }\; \frac{d^2 f}{dx^2} > 0,$$

within the domain of U. Comment on the graphical significance of these inequalities.

[6 marks]

2. A consumer has utility function

$$U(x, y) = \ln(xy + 1),$$

and is subject to the budget constraint

$$4x + 9y = 16,$$

where x and y denote the amounts of two commodities.
Determine the amounts purchased.
Sketch the budget constraint line and the indifference curve $U(x, y) = \ln(\frac{25}{9})$.

[7 marks]

3. The production function for a single commodity is given by

$$q(x_1, x_2) = \frac{x_1 x_2 + 1}{x_1 + 2x_2},$$

where $x_1 > 2$ and $x_2 > 1$ are the quantities of the two inputs used, per unit time. Sketch the isoquant $q = 1$.
Sketch also the production curve for the first commodity, if $x_2 = 1$.
Comment on the sign of the gradient of this curve.

[6 marks]

4. The total cost function for a single-commodity firm is

$$C(q) = q^3 - 6q^2 + 16q + 8,$$

where q is the quantity of the commodity produced in unit time.
Determine

(i) The fixed cost;
(ii) The marginal cost function, $MC(q)$;
(iii) The average variable cost function $AVC(q)$.

Verify that $MC(q) > 0$ for $q \geq 0$, and explain the significance of this result.
Find the price at which the firm would be forced to cease production in the *short-run* in an ideal competitive market.

[8 marks]

5. In an ideal competitive market for one good, the weekly supply function is given by

$$S(p) = \frac{25p^2}{6 + p^2},$$

where p is the price per unit of the good, and the demand function is

$$D(p) = 14 - 2p,$$

for $p < 6$, where p is the price per unit good.
Determine the equilibrium price, and the amount sold in a week.

[4 marks]

6. A monopolist has cost function

$$C(q) = q^3 - 6q^2 + 16q + 8,$$

for the production of a quantity q of a good, per unit time. Write down the profit function, as a function of q, when the demand function is

$$D(p) = 13 - p, \quad p < 13.$$

Determine the market price. Verify that your answer maximises the monopolist's
profit function for $q \geq 0$.

[7 marks]

7. The population density, $n(t)$, of a species of fish satisfies

$$\frac{dn}{dt} = 6n - n^2.$$

Find the equilibrium densities. If the initial density is 3, to which equilibrium does the population tend?

[5 marks]

8. A mathematical model of the behaviour of two interacting species X and Y is described by the coupled differential equations

$$\frac{dx}{dt} = x(7-2x) - 3xy, \quad \frac{dy}{dt} = y(3-y) - xy,$$

where $x(t)$ and $y(t)$ are the population densities of X and Y, respectively. Find the equilibria - you are not required to classify them.

[6 marks]

9. The community matrix at a particular equilibrium point of a two-species model has distinct eigenvalues of the same sign. Describe briefly, with the aid of diagrams, the possible local behaviour of trajectories near to the equilibrium point.

[6 marks]

SECTION B

10. A consumer has preferences for two commodities quantified by the utility function

$$U(x, y) = (2\sqrt{x} + \sqrt{y})^2,$$

where x and y denote amounts of the two commodities X and Y, respectively.

Find an expression for the ratio

$$R(x, y) = -\frac{\partial U}{\partial x} / \frac{\partial U}{\partial y},$$

in terms of x and y. Explain the geometrical and economic interpretation of the function R.

Unit quantities of X and Y cost p and q, respectively, and the consumer has B to spend each week.

Find the demand functions for the two commodities, in terms of B, p and q. Deduce that if p increases, whilst B and q are held at constant values, then the expenditure on Y also increases.

If $q = 2$ and $B = 5$, plot the demand function for X, as a function of p.

[15 marks]

11. The total cost function of a publishing firm is

$$C(q) = q^3 - 7q^2 + 16q + 90,$$

where q is the number of books produced in unit time.
Show that, in an ideal competitive market, the firm would stop publishing in the short-run if $p < \frac{15}{4}$. Determine the supply function and sketch it.
The firm trades alongside $N - 1$ other identical firms in a market where the aggregate demand function is $2N(b-p)$. Find the equilibrium price and the number of books sold by each firm when (i) $b = 10$, (ii) $b = 4$.

[15 marks]

12. Two companies make the same good. Their cost functions are :

$$C_1(q_1) = 2q_1^2 + 3q_1 + 3,$$

$$C_2(q_2) = q_2^2 + 7q_2 + 2,$$

where q_1 and q_2 represent the quantities of the good produced by the two companies per hour.
The two companies, acting as a Cournot duopoly in a market with demand function $D(p) = 17 - p$, try to maximise their profits independently of each other. Show that the optimal outputs are $q_1 = q_2 = 2$ units per hour. Find the price charged per unit of the good and the profits made by the companies.
The firms tentatively agree to cooperate and maximise their joint profit. Find what the individual profits would be, if such an agreement was carried out.
However, the second company decides to trade elsewhere, leaving the first as a monopoly. Show that the latter is the worst of the three market scenarios for the consumer.

[15 marks]

13. The population density of snakes in a well defined region of a wild life reserve satisfies

$$\frac{dn}{dt} = 15n^2 - 3n^3.$$

Find the non-zero equilibrium density, and determine its stability.
A family of mongooses preys upon the snakes. The growth law for the population density of snakes becomes modified:

$$\frac{dn}{dt} = 15n^2 - 3n^3 - kn,$$

where k is the number of mongooses hunting snakes each day. Given that $n \leq 18$, find the new equilibrium population densities. Identify the density $n_s \neq 0$ at which the population is in stable equilibrium.
Explain what would happen if more than 18 mongooses hunted snakes each day.
Assuming $n \leq 18$, and the number of snakes caught each day by each mongoose is proportional to n_s, find how many mongooses would be required to ensure the largest daily catch.

[15 marks]

14. The population densities of two interacting species evolve with time according to the coupled equations :

$$\frac{dx}{dt} = x(6-x) - 2xy, \quad \frac{dy}{dt} = -\frac{5}{4}y + \frac{5}{8}xy,$$

where $x(t)$ and $y(t)$ represent the population densities of the two species. Comment on the biological significance of each of the terms on the right hand sides of these equations.
Show that there are equilibria at $(0,0), (6,0)$, and $(2,2)$. and classify them.
Linearise the growth equations in the neighbourhood of the coexistence equilibrium, $(2,2)$, using the substitutions $x = 2 + \epsilon_x$, $y = 2 + \epsilon_y$, where the deviations from the equilibrium are small. Hence show that the particular solutions corresponding to the initial conditions $x(0) = 2$, $y(0) = 2 + \delta$, where δ is small, are given approximately by

$$x(t) = 2 - 2\delta e^{-t}\sin(2t), \quad y(t) = 2 + \frac{\delta}{2}e^{-t}(\sin(2t) + 2\cos(2t)).$$

[15 marks]