PAPER CODE NO.
MATH225

THE UNIVERSITY *of* LIVERPOOL

JANUARY 2006 EXAMINATIONS

Bachelor of Science : Year 2
Master of Mathematics : Year 2

VECTOR CALCULUS WITH APPLICATIONS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Candidates should attempt all questions in Section A and three questions in Section B. Section A carries 55 % of the available marks. A formula sheet is attached at the end of the paper.

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SECTION A

1. Describe the following loci and find the corresponding equations in spherical coordinates (ρ, θ, ϕ), where $\rho \geq 0,\ 0 \leq \theta < 2\pi,\ 0 \leq \phi \leq \pi$:

(a) $x^2 + y^2 + z^2 = 4x$; [3 marks]

(b) $x^2 + y^2 = 1,\ z = \sqrt{x^2 + y^2}$; [3 marks]

(c) $y = x,\ z = 0$. [3 marks]

2. Find the Cartesian equations of (a) the tangent plane and (b) the normal line to the surface $x^2 + y^2 - xyz = 7$ at the point $P(2, 3, 1)$.

[8 marks]

3. Show that the vector field

$$\mathbf{F} = yze^{xyz}\mathbf{i} + xze^{xyz}\mathbf{j} + xye^{xyz}\mathbf{k}$$

is conservative. Find a scalar function $\varphi(x, y, z)$ such that $\mathbf{F} = \mathrm{grad}\varphi$.

[7 marks]

4. Given that $r = \sqrt{x^2 + y^2}$ and $f = f(r)$, show that

$$\mathrm{div}\left(\frac{f(r)}{r^2}(x\mathbf{i} + y\mathbf{j})\right) = \frac{f'(r)}{r}, \quad r \neq 0.$$

[6 marks]

5. A position-dependent force $\mathbf{F}(x, , y, z) = x\mathbf{i} - z\mathbf{j} - xy\mathbf{k}$ Newtons acts on a particle which, at time t seconds, has position vector $\mathbf{r}$ metres, where $\mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$. Find the work done by the force as the particle moves from the point $A(1, 0, 0)$ to the point $B(-1, 0, 6\pi)$.

[6 marks]

6. At time t the velocity $\mathbf{V}$ of a fluid at the point (x, y) is given by

$$\mathbf{V} = -\alpha x^2 t\mathbf{i} + \beta y t\mathbf{j},$$

where α and β are dimensional constants.

(a) Find the parametric equations of the curve traced out by the fluid particle which was initially at the point $2\mathbf{i} + \mathbf{j}$. Hence, by eliminating the parameter time, write down the equation of the pathline for this particle.

[4 marks]

(b) Find the equation that represents the family of streamlines.

[3 marks]

7. A solid G occupies the region which lies between the plane $z = x$ and the surface $z = x^2$, and the planes $y = 0$ and $y = 3$. The density $\rho(x, y, z)$ of the solid is given by

$$\rho(x, y, z) = \alpha(1 - z),$$

where α is a dimensional constant.

(i) Find the mass M of the solid G.

[7 marks]

(ii) Find the volume V of the solid G.

[5 marks]

SECTION B

8. The velocity V and the density ρ of a fluid at a point (x, y, z) and time t are given by

$$\mathbf{V} = xt^2\mathbf{i} + 2y\cos t\,\mathbf{j} - (2zt^2 - 5y^2\sin t)\mathbf{k}, \quad \rho = xf(t),$$

where $f(t)$ is an unknown function. Given that the equation of continuity is satisfied, that is,

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\mathbf{V}) = 0,$$

and that $f(0) = \rho_0$, find $f(t)$.

[6 marks]

Show that the continuity equation can be written in the form

$$\frac{D\rho}{Dt} + \rho\ \mathrm{div}\mathbf{V} = 0.$$

[**Hint:** You may assume the equality $\mathrm{div}(\varphi\mathbf{F}) = \varphi\ \mathrm{div}\mathbf{F} + \mathbf{F}\cdot\mathrm{grad}\varphi$.]

Hence, or otherwise, evaluate the convective derivative $\frac{D\rho}{Dt}$.

[5 marks]

Find $\frac{D\mathbf{V}}{Dt}$, the acceleration field. [4 marks]

9. State the Divergence Theorem for a vector field $\mathbf{F}$ defined for a solid G enclosed by a surface σ.

[2 marks]

Evaluate by direct integration the integral $\int\int_\sigma \mathbf{F}\cdot\mathbf{n}\ \mathrm{dS}$ where

$$\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k},$$

$\mathbf{n}$ is the outward unit normal, and σ is the surface of the region cut from the first octant $x \geq 0,\ y \geq 0,\ z \geq 0$ by the cylindrical surface $x^2 + y^2 = 1$ and the plane $z = 4$.

Verify your result using the Divergence Theorem.

[13 marks]

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10. State Stokes' Theorem for a vector field $\mathbf{F}$ defined on a surface σ bounded by a closed curve C.

[2 marks]

Evaluate by direct integration the integral $\int\int_\sigma (\text{curl}\mathbf{F}) \cdot \mathbf{n}\, dS$, where

$$\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k},$$

and σ is the surface $z = 1 - x^2$ which lies between the planes $x = 0$, $x = 1$ and $y = -2$, $y = 2$, and is oriented by upward unit normals.

Verify your result using Stokes' Theorem.

[13 marks]

11. Consider a two-dimensional fluid flow whose velocity

$$\mathbf{V}(x, y) = u(x, y)\mathbf{i} + v(x, y)\mathbf{j}$$

is given in terms of a potential function φ by the equation $\mathbf{V} = \text{grad}\varphi$. Show that if the flow is incompressible then the velocity potential φ satisfies the Laplace equation $\nabla^2\varphi = 0$.

[3 marks]

A river flows across the xy-plane in the positive direction of the x-axis and around a circular rock of radius 1 centred at the origin. The flow is described by the velocity potential φ which outside the circular rock is given by

$$\varphi(x, y) = x + \frac{x}{x^2 + y^2} \quad \text{for} \quad x^2 + y^2 \geq 1.$$

Find the velocity $\mathbf{V}$ of the flow. Hence show that $\text{div}\mathbf{V} = 0$.

[4 marks]

Write down the expressions for the velocity components in terms of a stream function $\psi(x, y)$. Find the corresponding stream function ψ.

[5 marks]

Show that the velocity $\mathbf{V}$ of the flow is tangent to the circle $x^2+y^2 = 1$. This means that no water crosses the circle. The water on the outside must therefore all flow around the circle.

[3 marks]

12. Derive the equation of motion for an inviscid fluid.

[7 marks]

Show that in the static case ($\mathbf{V} = \mathbf{0}$), the equation of motion becomes

$$\mathbf{F} = \frac{1}{\rho} \operatorname{grad} p, \tag{1}$$

where $\mathbf{F}$ is the body force acting on the fluid, p is the pressure, and ρ is the fluid density.

[3 marks]

Consider a column of incompressible fluid oriented vertically parallel to the z-axis. Assuming that the only external force acting on the fluid is the downward uniform gravitational force $\mathbf{F}$, apply equation (1) to show that

$$p = p_0 - \rho g z,$$

where g is the acceleration due to gravity, and p_0 is a constant.

[5 marks]

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Formula Sheet

Spherical Polar Coordinates (ρ, θ, ϕ)

$$\text{grad}G = \frac{\partial G}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho \sin\phi}\frac{\partial G}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial G}{\partial \phi}\mathbf{e}_\phi$$

$$\text{div}\mathbf{F} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 F_\rho) + \frac{1}{\rho \sin\phi}\frac{\partial}{\partial \theta}(F_\theta) + \frac{1}{\rho \sin\phi}\frac{\partial}{\partial \phi}(\sin\phi F_\phi)$$

$$\text{curl}\mathbf{F} = \frac{1}{\rho^2 \sin\phi}\begin{vmatrix} \mathbf{e}_\rho & \rho\sin\phi\mathbf{e}_\theta & \rho\mathbf{e}_\phi \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_\rho & \rho\sin\phi F_\theta & \rho F_\phi \end{vmatrix}$$

Cylindrical Polar Coordinates (r, θ, z)

$$\text{grad}G = \frac{\partial G}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial G}{\partial \theta}\mathbf{e}_\theta + \frac{\partial G}{\partial z}\mathbf{e}_z$$

$$\text{div}\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(F_\theta) + \frac{\partial}{\partial z}(F_z)$$

$$\text{curl}\mathbf{F} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}$$

Equation of motion of an inviscid fluid

$$\frac{\partial \mathbf{V}}{\partial t} + \frac{1}{2}\,\text{grad}(\mathbf{V}^2) - \mathbf{V} \times \text{curl}\mathbf{V} = \mathbf{F} - \frac{1}{\rho}\,\text{grad}\,p$$

or

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \text{grad})\mathbf{V} = \mathbf{F} - \frac{1}{\rho}\,\text{grad}\,p$$