SECTION A

1. *Find the general solution of the linear ordinary differential equation*

$$\frac{dy}{dx} + \tan(x)\, y = 2x\cos(x) \ ,$$

leaving your answer in the form $y = f(x)$.

[4 marks]

Solution [Homework]

The integrating factor is

$$\mu(x) = e^{\int \tan(x)\, dx} = e^{-\log(\cos(x))} = 1/\cos(x).$$

Therefore the equation becomes

$$\frac{1}{\cos(x)}\frac{dy}{dx} + \frac{\sin(x)}{\cos^2(x)} y = \frac{d}{dx}\left(\frac{y}{\cos(x)}\right) = 2x$$

Therefore

$$\frac{y}{\cos(x)} = x^2 + A,$$

so the solution is

$$y(x) = A\cos(x) + x^2\cos(x) \ .$$

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2. *Solve the initial value problem*

$$\frac{dy}{dx} = \frac{3x + y}{x} \ ; \qquad y(1) = 2 \ .$$

[5 marks]

Solution [Homework]

This is a homogeneous equation. The substitution $y(x) = xv(x)$ gives

$$x\,\frac{dv}{dx} + v = 3 + v$$

so

$$\frac{dv}{dx} = 3/x$$

so

$$v(x) = 3\ln(x) + c$$

and the general solution is

$$y(x) = 3x\ln(x) + cx$$

Putting $x = 1$ gives $y(1) = c$, so $c = 2$ and the solution is

$$y(x) = 3x\ln(x) + 2x \ .$$

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3. *Find the general solution of the following system of equations:*

$$\begin{aligned} \frac{dx}{dt} &= x - 2y \ , \\ \frac{dy}{dt} &= 5x + 3y \ . \end{aligned}$$

[9 marks]

Solution [Homework]
In vector form, we have

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} \quad \text{where} \quad A = \begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix} \ .$$

The characteristic polynomial is

$$\lambda^2 - 4\lambda + 13 = 0$$

so

$$\lambda = 2 \pm 3i.$$

The eigenvector $\mathbf{v}$ corresponding to $\lambda = 2 + 3i$ is

$$\mathbf{v} = \begin{pmatrix} 2 \\ -1 - 3i \end{pmatrix}$$

Hence the general solution is

$$\mathbf{y}(t) = Ae^{2t} \begin{pmatrix} 2\cos(3t) \\ -\cos(3t) + 3\sin(3t) \end{pmatrix} + Be^{2t} \begin{pmatrix} 2\sin(3t) \\ -\sin(3t) - 3\cos(3t) \end{pmatrix} \ ,$$

or, componentwise,

$$\begin{aligned} x(t) &= 2Ae^{2t}\cos(3t) + 2Be^{2t}\sin(3t) \ , \\ y(x) &= Ae^{2t}\left(-\cos(3t) + 3\sin(3t)\right) - Be^2t\left(\sin(3t) + 3\cos(3t)\right) \ . \end{aligned}$$

This question may also be solved by substitution or using the Laplace transform.

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4. *The Laplace transform of a function $f(t)$ is defined by*

$$\mathcal{L}\left\{f(t)\right\} = \widetilde{f}(s) = \int_0^\infty f(t)e^{-st}t)\ dt\ .$$

(i) *Show that*

$$\mathcal{L}\left\{tf(t)\right\} = -\frac{d\widetilde{f}}{ds}\ .$$

[3 marks]

Solution [Bookwork]

$$\begin{aligned}\mathcal{L}\left\{tf'(t)\right\} &= \int_0^\infty f(t)te^{-st}\ dt \ = \ \int_0^\infty -f(t)\frac{d}{ds}\left(e^{-st}\right)\ dt \\ &= -)\frac{d}{ds}\int_0^\infty f(t)e^{-st}\ dt \ = \ -\frac{d\widetilde{f}}{ds}\ .\end{aligned}$$

(ii) *Compute the Laplace transform of $t^2\sin(3t)$.*

[5 marks]

Solution [Bookwork]

The Laplace transform of $t^2\sin(3t)$ is second derivative of the Laplace transform of $\sin(3t)$, so

$$\begin{aligned}\mathcal{L}\left\{t^2\sin(3t)\right\} &= \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) \\ &= \frac{d}{ds}\left(\frac{-6s}{(s^2+9)^2}\right) \\ &= \frac{-6}{(s^2+9)^2} + \frac{24}{(s^2+9)^3} \\ &= \frac{-6s^2-30}{(s^2+9)^3}\ .\end{aligned}$$

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5. *Calculate the Fourier cosine series of period π for the function $f(x)$ defined for $0 \le x \le \pi$ by*

$$f(x) = \sin(x) .$$

Hint: *For any A and B,*

$$\sin(A)\cos(B) = \tfrac{1}{2}\left(\sin(A+B) + \sin(A-B)\right) .$$

[7 marks]

Solution [Homework]

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^\pi f(x)\,dx \\
&= \frac{2}{\pi}\int_0^\pi \sin(x)\,dx \;=\; \frac{2}{\pi}\left[-\cos(x)\right]_0^\pi \;=\; \frac{2}{\pi}\big(1-\cos(\pi)\big) \;=\; 4/\pi \\
a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx \\
&= \frac{2}{\pi}\int_0^\pi \sin(x)\cos(nx)\,dx \;=\; \frac{2}{\pi}\int_0^\pi \frac{1}{2}\big(\sin((1+n)x)+\sin((1-n)x)\big)\,dx \\
&= \frac{1}{\pi}\left[\frac{-\cos((1+n)x)}{1+n} + \frac{-\cos((1-n)x)}{1-n}\right]_0^\pi \;=\; \frac{1}{\pi}\left(\frac{1+(-1)^n}{1+n} + \frac{1+(-1)^n}{1-n}\right) \\
&= \frac{2\,(1+(-1)^n)}{\pi(1-n^2)}
\end{aligned}$$

So the Fourier cosine series is

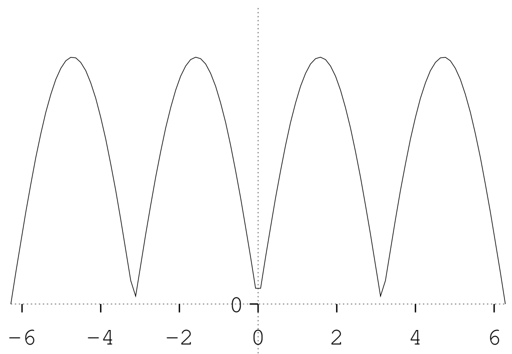
$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) \;=\; \frac{4}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(1+(-1)^n)}{1-n^2}\cos(nx) \\
&= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{1-4n^2}\cos(2nx) .
\end{aligned}$$

Note: Full marks can be obtained without noticing that $a_1 = 0/0$.

Sketch the graph of this cosine series for $-2\pi < x < 2\pi$.

[2 marks]

Solution [Bookwork]



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6. *The Cauchy-Riemann equations for the real and imaginary parts* $u(x,y)$ *and* $v(x,y)$ *of a complex function* $f(x+iy)$ *are*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \, .$$

(i) *Suppose* $v(x,y)$ *is given by* $v(x,y) = 3x^2y - y^3 + x$. *Find a function* $u(x,y)$ *so that* u *and* v *satisfy the Cauchy-Riemann equations.*

[6 marks]

Solution:

Integrating the equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

we find

$$u(x,y) = x^3 - 3xy^2 + k(y)$$

or some function k. Then

$$\frac{\partial u}{\partial y} = -6xy + k'(y) = -6xy - 1 = -\frac{\partial v}{\partial x}$$

so $k(y) = -y$ (plus an arbitrary constant). Therefore

$$u(x,y) = x^3 - 3xy^2 - y.$$

(ii) *Find a function* $f(z)$ *such that* $f(x+iy) = u(x,y) + iv(x,y)$.

[3 marks]

Solution [Class]

We have $f(x+iy) = x^3 - 3xy^2 - y + i(3x^2y - y^3 + x)$. Taking $x = z$ and $y = 0$ we find

$$f(z) = z^3 + iz.$$

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7. *The function* $u(x,t) = F(x)\cos(\lambda ct)$*, where* c *and* λ *are positive constants, is a nontrivial solution of the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \, .$$

Show that $F(x)$ *satisfies the ordinary differential equation*

$$F'' + \lambda^2 F = 0 \, .$$

[4 marks]

Solution [Bookwork]
Substituting into the wave equation gives

$$-\lambda^2 c^2 F(x)\cos(\lambda ct) = c^2 F''(x)\cos(\lambda ct) \, .$$

Cancel $\cos(\lambda ct)$ from each side to give

$$-\lambda^2 F(x) = F''(x)$$

which implies

$$F''(x) + \lambda^2 F(x) = 0.$$

Given that u *also satisfies the boundary conditions*

$$u(0,t) = u(L,t) = 0 \, ,$$

show that the possible values of λ *are* $n\pi/L$*, where* n *is a positive integer, and find the corresponding functions* $F(x)$*.*

[4 marks]

Solution [Bookwork]
The general solution for $F(x)$ is

$$F(x) = A\sin(\lambda x) + B\cos(\lambda x).$$

The boundary conditions give $F(0) = F(L) = 0$. Now

$$F(0) = A\sin(0) + B\cos(0) = B = 0$$

and then

$$F(L) = A\sin(\lambda L) = 0$$

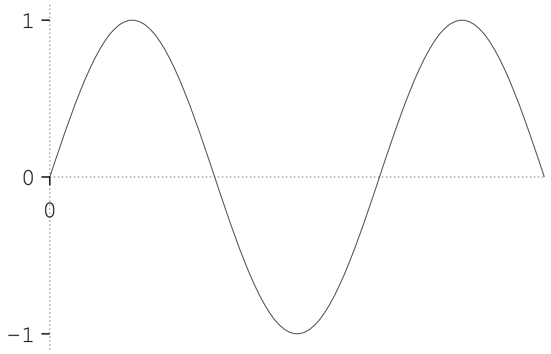
so either $A = 0$ or $\sin(\lambda L) = 0$. $A = 0$ gives the trivial solution, so $\sin(\lambda L) = 0$ which means $\lambda L = n\pi$ or $\lambda = n\pi/L$ for some positive integer n. Thus

$$\lambda = n\pi/L \quad \text{ and } F_n(x) = \sin\left(\frac{n\pi x}{L}\right) \, .$$

Sketch $F(x)$ on the interval $0 \leq x \leq L$ for $n = 3$.

[2 marks]

Solution [Bookwork]



SECTION B

8. *Find the solution of the following ordinary differential equation, using the initial condition* $y(1) = y'(1) = 0$.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 9y = 27\ln(x) - 4x^2 \; .$$

[15 marks]

Solution [Homework]

Try $y_c(x) = x^m$ as a solution to the complementary equation. We obtain the auxiliary equation

$$m(m-1) - 5m + 9 = 0$$

which has a double root $m = 3$. Hence the complementary function is

$$y_c(x) = Ax^3 + Bx^3\ln(x) \, .$$

[6 marks for this part]

Try particular integral $y_p(x) = a\ln(x) + b + cx^2$. Substituting into the equation gives

$$x^2(-a/x^2 + 2c) - 5x(a/x + 2cx) + 9(a\ln(x) + b + cx^2) = 27\ln(x) - 4x^2$$

and simplifying gives

$$9a\ln(x) + (9b - 6a) + cx^2 = 27\ln(x) - 4x^2$$

from which we deduce $a = 3$, $b = 2$ and $c = -4$. Hence the general solution is

$$y(x) = Ax^3 + Bx^3\ln(x) + 3\ln(x) - 4x^2 + 2 \, .$$

[6 marks for this part]

The derivative of $y(x)$ is

$$y'(x) = 3Ax^2 + Bx^2(3\ln(x) + 1) + 3/x - 8x,$$

so

$$y(1) = A - 2 \quad \text{and} \quad y'(1) = 3A + B - 5.$$

Therefore $A = 2$ and $B = -1$, so

$$y(x) = 2x^3 - x^3\ln(x) + 3\ln(x) - 4x^2 + 2 \, .$$

[3 marks for this part]

(This problem can also be solved by using the substitution $x = e^t$.)

9.

(a) *Find a function $h(t)$ such that the solution of the ordinary differential equation*

$$\frac{d^2y}{dt^2} + 4y = g(t)$$

with initial conditions $y(0) = y'(0) = 0$ is given by the convolution integral

$$\int_0^t g(t-\tau)h(\tau)\,d\tau\ .$$

[6 marks]

Solution:

Take the Laplace transform of both sides to obtain

$$s^2Y(s) + 4Y(s) = G(s).$$

Rearranging gives

$$Y(s) = \frac{1}{s^2+4}G(s)$$

so $h(t)$ is the inverse Laplace transform of $1/(s^2+4)$, which gives

$$h(t) = \tfrac{1}{2}\sin(2t).$$

(b) *Show that the Fourier series of the 2π-periodic odd function $f(x)$ defined by $f(x) = x$ for $-\pi \leq x \leq \pi$ is*

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(nx)}{n}\ .$$

By evaluating the square integral of $f(x)$ and of its Fourier series, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\ .$$

[9 marks]

Solution:

The Fourier coefficients b_n are given by

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi x\sin(nx)\,dx = \frac{2}{\pi}\left[\frac{-x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^\pi \\ &= \frac{2}{\pi}\,\frac{-\pi(-1)^n}{n} = \frac{2(-1)^{n+1}}{n} \end{aligned}$$

The square integral of $f(x)$ over $[-\pi,\pi]$ is

$$\int_{-\pi}^{\pi} \left(f(x)\right)^2\,dx = \int_{-\pi}^{\pi} x^2\,dx = \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3}\,.$$

The square integral of the Fourier sine series gives

$$\int_{\pi}^{\pi} \left(\sum_{n=1}^{\infty} b_n \sin(nx) \right)^2 \, dx = \pi \sum_{n=1}^{\infty} b_n^2 \, .$$

Taking $b_n = 2(-1)^{n+1}/n$ gives

$$\frac{2\pi^3}{3} = \int_{-\pi}^{\pi} x^2 \, dx = \pi \sum_{n=1}^{\infty} \frac{4}{n^2} = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2} \, ,$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^3}{3} \frac{1}{4\pi} = \frac{\pi^2}{6} \, .$$

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10. *Show that the characteristic curves of the first-order partial differential equation*

$$2\frac{\partial u}{\partial x} - xy\frac{\partial u}{\partial y} = u$$

are given by

$$x(t) = x_0 + 2t, \quad y(t) = y_0 e^{-(x_0 t + t^2)}.$$

[6 marks]

Solution [Homework]

The characteristic equations are

$$\frac{dx}{dt} = 2 \quad \text{and} \quad \frac{dy}{dt} = -xy\,.$$

Solving first for x, we find

$$x = x_0 + 2t,$$

so

$$\frac{dy}{dt} = -(x_0 + 2t)y.$$

Solving this by separation of variables we find

$$\ln(y) = -(x_0 t + t^2) + \ln(y_0)$$

so

$$y = y_0 e^{-(x_0 t_t^2)}\,.$$

By considering the boundary value problem $u(0, s) = f(s)$, or otherwise, find the general solution of this equation.

[9 marks]

Solution [Homework]

The right-hand side gives

$$\frac{du}{dt} = u$$

so

$$u = u_0 e^t\,.$$

The boundary conditions give $x_0 = 0$, $y_0 = s$ and $u_0 = f(s)$, so

$$x = 2t, \qquad y = se^{-t^2} \quad \text{and} \quad u = f(s)e^t\,.$$

Therefore

$$t = x/2 \quad \text{and } s = ye^{t^2} = ye^{x^2/4}\,,$$

so the general solution is

$$u(x, y) = f(ye^{x^2/4})e^{x/2}\,.$$

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11. *Write down the general solution of the heat equation*

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

in a bar of length L, whose left and right hand ends are held at temperatures T_0 and T_1 respectively.

[4 marks]

Solution [Bookwork]

$$u(x,t) = T_0 + \frac{x}{L} T_1 + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2} \;.$$

Find the particular solution of the heat equation in a bar for which the initial temperature distribution is

$$u(x,0) = \begin{cases} 0\,^\circ\mathrm{C} & \textit{if} \;\; 0 < x < L/2 \\ 50\,^\circ\mathrm{C} & \textit{if} \;\; L/2 < x < L \end{cases}$$

and the ends are held at 20 °C.

[11 marks]

Solution [Homework]

The equilibrium solution is $u_e(x) = 20\,^\circ\mathrm{C}$. Let $f(x) = u(x,0) - u_e(x)$. The coefficients of the Fourier sine series of $f(x)$ are

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^{L/2} -20\sin\left(\frac{n\pi x}{L}\right)\,dx + \frac{2}{L}\int_{L/2}^{L} 30\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\left[\frac{20L\cos(n\pi x/L)}{n\pi}\right]_0^{L/2} + \frac{2}{L}\left[\frac{-30L\cos(n\pi x/L)}{n\pi}\right]_{L/2}^{L} \\ &= \frac{40\,(\cos(n\pi/2)-1)}{n\pi} + \frac{60\,(\cos(n\pi)-\cos(n\pi/2))}{n\pi} \\ &= \frac{3\cos(n\pi)-\cos(n\pi/2)-2}{n\pi}\,20\,^\circ\mathrm{C} \end{aligned}$$

Therefore,

$$u(x,t) = 20\,^\circ\mathrm{C}\left(1 + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{3\cos(n\pi)-\cos(n\pi/2)-2}{n}\sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2\kappa t/L^2}\right)$$

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12. *Show that the function*

$$u(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\big(A_n \cosh(n\pi x/L) + B_n \sinh(n\pi x/L)\big)$$

satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the square $0 \le x, y \le L$ *with boundary conditions* $u(0,y) = u(L,y) = 0$.

[5 marks]

Solution [Example]

Find the particular solution for which

$$u(x,0) = 0 \quad \text{and} \quad u(x,L) = x(L-x).$$

[10 marks]

Solution [Homework]

The boundary condition $u(x,0) = 0$ implies

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = 0$$

so the coefficients A_n are all 0.

The boundary condition $u(x,L) = 1$ gives

$$\sum_{n=1}^{\infty} B_n \sinh(n\pi) \sin\left(\frac{n\pi x}{L}\right) = 1\,.$$

The Fourier coefficients $b_n = B_n \sinh(n\pi)$ are then given by

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\left[\frac{-Lx(L-x)\cos(n\pi x/L)}{n\pi} + \frac{L^2(L-2x)\sin(n\pi x/L)}{n^2\pi^2} - \frac{2L^3\cos(n\pi x/L)}{n^3\pi^3}\right]_0^L \\
&= \frac{2}{L}\,\frac{2L^3(1-(-1)^n)}{n^3\pi^3} \;=\; \frac{4L^2}{n^3\pi^3}\big(1-(-1)^n\big)\,.
\end{aligned}$$

Thus

$$u(x,y) = \frac{4L^2}{\pi^3}\sum_{n=1}^{\infty} \frac{\big(1-(-1)^n\big)}{n^3\sinh(n\pi)} \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$
