PAPER CODE NO.
MATH224

THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 2002 EXAMINATIONS

Bachelor of Arts : Year 2
Bachelor of Science : Year 1
Bachelor of Science : Year 2
Bachelor of Science : Year 3
Master of Mathematics : Year 2
Master of Physics : Year 2

INTRODUCTION TO THE METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two hours and a half

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B. The marks shown against the questions, or parts of questions, indicate their relative weights. The total of marks available in Section A is 55.

SECTION A

1. Find the general solution of the linear ordinary differential equation

$$\frac{dy}{dx} + \tan(x)\, y = 2x\cos(x) \ ,$$

leaving your answer in the form $y = f(x)$.

[4 marks]

2. Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x+y}{x} \ ; \qquad y(1) = 2 \ .$$

[5 marks]

3. Find the general solution of the following system of equations:

$$\begin{aligned} \frac{dx}{dt} &= x - 2y \ , \\ \frac{dy}{dt} &= 5x + 3y \ . \end{aligned}$$

[9 marks]

4. The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}\left\{f(t)\right\} = \widetilde{f}(s) = \int_0^\infty f(t)e^{-st}t)\ dt \ .$$

(i) Show that

$$\mathcal{L}\left\{tf(t)\right\} = -\frac{d\widetilde{f}}{ds} \ .$$

[3 marks]

(ii) Compute the Laplace transform of $t^2\sin(3t)$.

[5 marks]

5. Calculate the Fourier cosine series of period π for the function $f(x)$ defined for $0 \le x \le \pi$ by

$$f(x) = \sin(x)\,.$$

Hint: For any A and B,

$$\sin(A)\cos(B) = \tfrac{1}{2}\left(\sin(A+B) + \sin(A-B)\right).$$

[7 marks]

Sketch the graph of this cosine series for $-2\pi < x < 2\pi$.

[2 marks]

6. The Cauchy-Riemann equations for the real and imaginary parts $u(x,y)$ and $v(x,y)$ of a complex function $f(x+iy)$ are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,.$$

(i) Suppose $v(x,y)$ is given by $v(x,y) = 3x^2y - y^3 + x$. Find a function $u(x,y)$ so that u and v satisfy the Cauchy-Riemann equations.

[6 marks]

(ii) Find a function $f(z)$ such that $f(x+iy) = u(x,y) + iv(x,y)$.

[3 marks]

7. The function $u(x,t) = F(x)\cos(\lambda ct)$, where c and λ are positive constants, is a nontrivial solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}\,.$$

Show that $F(x)$ satisfies the ordinary differential equation

$$F'' + \lambda^2 F = 0\,.$$

[4 marks]

Given that u also satisfies the boundary conditions

$$u(0,t) = u(L,t) = 0\,,$$

show that the possible values of λ are $n\pi/L$, where n is a positive integer, and find the corresponding functions $F(x)$.

[4 marks]

Sketch $F(x)$ on the interval $0 \le x \le L$ for $n = 3$.

[2 marks]

SECTION B

8. Find the solution of the following ordinary differential equation, using the initial condition $y(1) = y'(1) = 0$.

$$x^2 \frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 9y = 27\ln(x) - 4x^2 \; .$$

[15 marks]

9.

(a) Find a function $h(t)$ such that the solution of the ordinary differential equation

$$\frac{d^2y}{dt^2} + 4y = g(t)$$

with initial conditions $y(0) = y'(0) = 0$ is given by the convolution integral

$$\int_0^t g(t-\tau)h(\tau)\, d\tau \; .$$

[6 marks]

(b) Show that the Fourier series of the 2π-periodic odd function $f(x)$ defined by $f(x) = x$ for $-\pi \leq x \leq \pi$ is

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(nx)}{n} \; .$$

By evaluating the square integral of $f(x)$ and of its Fourier series, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \; .$$

[9 marks]

10. Show that the characteristic curves of the first-order partial differential equation

$$2\frac{\partial u}{\partial x} - xy\frac{\partial u}{\partial y} = u$$

are given by

$$x(t) = x_0 + 2t, \quad y(t) = y_0 e^{-(x_0 t + t^2)}.$$

[6 marks]

By considering the boundary value problem $u(0, s) = f(s)$, or otherwise, find the general solution of this equation.

[9 marks]

11. Write down the general solution of the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

in a bar of length L, whose left and right hand ends are held at temperatures T_0 and T_1 respectively.

[4 marks]

Find the particular solution of the heat equation in a bar for which the initial temperature distribution is

$$u(x, 0) = \begin{cases} 0\,°\text{C} & \text{if } 0 < x < L/2 \\ 50\,°\text{C} & \text{if } L/2 < x < L \end{cases}$$

and the ends are held at 20 °C.

[11 marks]

12. Show that the function

$$u(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \big(A_n \cosh\left(n\pi x/L\right) + B_n \sinh\left(n\pi x/L\right)\big)$$

satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the square $0 \le x, y \le L$ with boundary conditions $u(0,y) = u(L,y) = 0$.

[5 marks]

Find the particular solution for which

$$u(x,0) = 0 \quad \text{and} \quad u(x,L) = x(L-x).$$

[10 marks]