MATH224 September resit

Solutions

1. Separate the variables:

$$\begin{aligned}
\int_2^y (y+3)dy &= \int_0^x (1+x^2)dx \\
\left[y^2/2+3y\right]_2^y &= \left[x+x^3/3\right]_0^x \\
y^2/2+3y-2-6 &= x+x^3/3 \\
i.e. y^2+6y-16 &= 2x+2x^3/3 \\
(y+3)^2 &= 2x^3/3+2x+25 \\
y+3 = \sqrt{(2x^3+6x+75)/3} &
\end{aligned}$$

2. Put $y = x^m$: then

$$\begin{aligned}
m(m-1)-2m+2 &= 0 \\
\Rightarrow (m-2)(m-1) &= 0 \\
\Rightarrow y &= Ax+Bx^2
\end{aligned}$$

where A and B are constants.

3.

$$\begin{aligned}
y &= 8x - 5\frac{dx}{dt} \\
\Rightarrow 5\frac{dy}{dt} &= 40\frac{dx}{dt} - 25\frac{d^2x}{dt^2}, \\
\Rightarrow -6x &= 40\frac{dx}{dt} - 25\frac{d^2x}{dt^2} - 7(8x - 5\frac{dx}{dt}) \\
\Rightarrow 25\frac{d^2x}{dt^2} - 75\frac{dx}{dt} + 50x &= 0
\end{aligned}$$

This has the characteristic equation

$$r^{-}3r + 2 = 0 \Rightarrow r = 1, 2$$

Solution is

$$x = A \exp 2t + B \exp t.$$

Thus

$$\frac{dx}{dt} = 2A \exp 2t + B \exp t$$

so

$$y = -2A \exp(2t) + 3B \exp(t)$$

From the initial conditions, we have

$$\begin{aligned} x(0) &= 2 = A + B \\ y(0) &= 1 = -2A + 3B \\ \Rightarrow B &= 1 \\ A &= 1 \end{aligned}$$

Hence

$$\begin{aligned} x &= e^{2t} + e^{t} \\ y &= -2e^{2t} + e^{t} \end{aligned}$$

4. (i) From the definition of the L.T.

$$\begin{aligned} \mathcal{L}\{\exp -ktf(t)\} &= \int_0^\infty e^{-st} e^{-kt} f(t) dt \\ &= F(k+s) \end{aligned}$$

(ii)

$$\begin{aligned} \mathcal{L}\{tf(t)\} &= \int_0^\infty e^{-st} t f(t) dt \\ &= -\frac{d}{ds} \int \exp -st f(t) dt \\ &= \frac{dF(s)}{dx} \end{aligned}$$

(iii)

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st} dt \\
&= \frac{1}{s}
\end{aligned}$$

(ii) and (iii) imply

$$\begin{aligned}
\mathcal{L}\{t\} &= \frac{1}{s^2} \\
\mathcal{L}\{t^3\} &= \frac{d^2 L(t)}{ds^2} = \frac{6}{s^4}
\end{aligned}$$

Thus using (i)

$$\mathcal{L}^{-1}\{(s+3)^{-4}\} = \exp -3t \mathcal{L}^{-1}\{s^{-4}\} = \exp -3t t^3/6$$

5. We have

$$\begin{aligned}
u(x,t) &= F(x)G(t) \\
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{k}\frac{\partial u}{\partial t} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{d^2 F}{dx^2} G(t) \\
\frac{\partial u}{\partial t} &= \frac{dG}{dt} F(x)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{d^2 F}{\partial x^2} G(t) &= \frac{1}{k} F \frac{dG}{dt} \\
\Rightarrow \frac{1}{F}\frac{d^2 F}{dx^2} &= \frac{1}{kG}\frac{dG}{dt}
\end{aligned}$$

The LHS is independent of t and so the RHS must also be. The RHS is however independent of x and so is independent of both x and t and must therefore be a constant α. The equations for F and G are thus

$$\frac{d^2F}{dx^2} = \alpha F \qquad \text{and} \qquad \frac{dG}{dt} = k\alpha G$$

From the boundary conditions we have $u(0,t) = u(d,t) = 0$ so $F(0) = F(d) = 0$. Now if

$$F(x) = \sin\left(\frac{2\pi nx}{d}\right)$$

Then

$$\frac{F(x)}{dx^2} = \frac{-4\pi^2n^2}{d^2}\sin\left(\frac{2\pi nx}{d}\right) = \frac{-4\pi^2n^2}{d^2}F(x)$$

So that the above equation is satisfied by putting

$$\alpha = \frac{-4\pi^2n^2}{d^2}$$

The boundary conditions are satisfied as $\sin(2n\pi) = 0$

6.

$$\xi = x - y, \quad \eta = y$$

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \xi^2} \\
\frac{\partial^2 u}{\partial y^2} &= -\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta} \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} - 2\frac{\partial^2 u}{\partial \eta^2}
\end{aligned}$$

Therefore

$$u_{xx} + 2u_{xy} + u_{yy} = \frac{\partial^2 u}{\partial \xi^2} + 2\left(-\frac{\partial^2 u}{\partial \xi^2 +} + \frac{\partial^2 u}{\partial \xi \partial \eta}\right) + \frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} = \frac{\partial^2 u}{\partial \eta^2}$$

If this equals 0, we get

$$\frac{\partial^2 u}{\partial \eta^2} = 0$$

Hence

$$u = f(\xi)\eta + g(\xi) = f(x-y)y + g(x-y)$$

7. The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$u(x,y) = x^3 - 3xy^2 - y^2 + x^2$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 2x$$
$$\frac{\partial^2 u}{\partial x^2} = 6x + 2$$
$$\frac{\partial u}{\partial y} = -6xy - 2y$$
$$\frac{\partial^2 u}{\partial y^2} = -6x - 2$$

Laplace's equation in 2d is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which is therefore clearly satisfied.

From the first CR relation, we have

$$-\frac{\partial v}{\partial x} = -6xy - 2y$$
$$\Rightarrow v = 3x^2 y + 2yx + f(y)$$

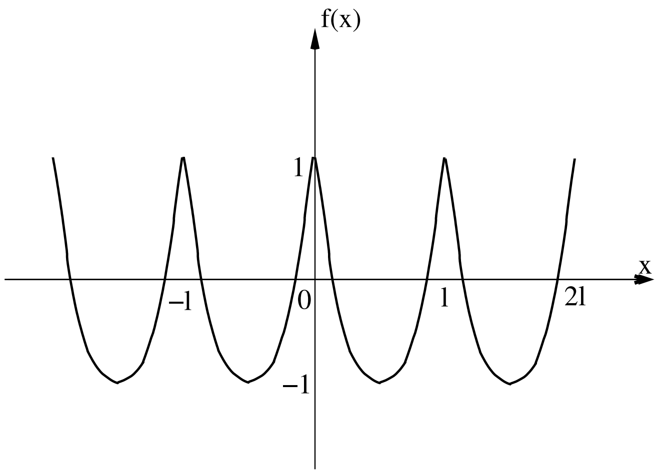
From the second CR relation, we have

$$\begin{aligned}
\frac{\partial v}{\partial y} &= 3x^2 + 2x + \frac{df}{dy} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 2x \\
\Rightarrow \frac{df}{dy} &= -3y^2 \\
\Rightarrow f(y) &= -y^3 + C \\
\Rightarrow v &= 3x^2 y + 2xy - y^3 + C
\end{aligned}$$

10 .

$$f(x) = 1 - 2\left|\sin\left(\frac{\pi x}{l}\right)\right| \qquad -l \le x << 0$$

The period is l (see graph)



Also, $b_n = 0$ since f(x) is even. From lectures, we have

$$a_0 = \frac{2}{l}\int_{-l/2}^{l/2} \left(1 - 2\left|\sin\left(\frac{\pi x}{l}\right)\right|\right) dx = 2 - \frac{8}{\pi}$$

$$\begin{aligned}
a_n &= \frac{2}{l}\int_{-l/2}^{l/2} \left(1 - 2\left|\sin\left(\frac{\pi x}{l}\right)\right|\right) \cos\left(\frac{2n\pi x}{l}\right) dx \\
&= \frac{4}{l}\int_{0}^{l/2} \left(1 - 2\sin\left(\frac{\pi x}{l}\right)\right) \cos\left(\frac{2n\pi x}{l}\right) \\
&= \frac{4}{l}\int_{0}^{l/2} \left\{\cos\left(\frac{2n\pi x}{l}\right) - \sin\left(\frac{(2n+1)\pi x}{l}\right) + \sin\left(\frac{(2n-1)\pi x}{l}\right)\right\} dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{\pi}\left[0+\frac{\cos\left(\frac{(2n+1)\pi}{2}\right)-1}{2n+1}-\frac{\cos\left(\frac{(2n-1)\pi}{2}\right)-1}{2n-1}\right] \\
&= \frac{4}{\pi}\left[\frac{-1}{2n+1}+\frac{1}{2n-1}\right] \\
&= \frac{8}{\pi(2n+1)(2n-1)} \\
&= \frac{8}{\pi(4n^2-1)}
\end{aligned}
$$